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Renormalization Group Recursion Formulas and Flows of 2D $O(N)$ Spin Models

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Renormalization group recursion formulas for classical $O(N)$ spin models in two dimensions are obtained. The main part of the recursion formulas is solved and yields the flows which are very close to those of the hierarchical model approximations of Dyson–Wilson type. Spontaneous mass generations also take place under our approximation.

KEY WORDS: $O(N)$ spin model; auxiliary field; block spin transformation; renormalization group flow.

1. INTRODUCTION

Though spontaneous mass generations in 2D non-Abelian sigma models are widely believed,[17, 19] we still do not have a rigorous proof. This problem is very much related to the long-standing problem of quark confinement in 4 dimensional (4D) non-Abelian lattice gauge theories. These models exhibit no phase transitions in the hierarchical model approximation of Wilson–Dyson type or Migdal–Kadanov type.[11]

One of main difficulties in these models is that the field variables form compact manifolds and the block spin transformations break the structures. In some cases, this can be avoided by introducing an auxiliary field ψ.[1, 2] Using this idea, together with the help of the cluster expansion,[3, 10, 18] we showed[14] that

$$\frac{\beta_c}{N} \geqslant \text{const} \log N \tag{1.1}$$

in the 2D $O(N)$ sigma model, where β_c^{-1} is the critical temperature.

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In this paper, we apply the exact block-spin transformation to the functional integral of the system, and extract the main part of the renormalization recursion formulas. Though the control of small non-local marginal terms are left to the future,[13] we show in this paper that

Main Theorem. (i) The main part of the renormalization group recursion formulas reproduces the renormalization group flow of the hierarchical model of Dyson–Wilson type,[4, 19] rather than that of Gallavotti type.[6, 9]

(ii) The recursion formulas converge to a massive state no matter how low the initial temperature is.

The recursion relations are derived by applying the standard block spin transformation of Wilson–Kadanoff type[19] to the $O(N)$ spin model (with large N) rewritten by the auxiliary field ψ. The block spin transformation was first formulated in a mathematically rigorous way by Gawedzki and Kupiainen,[7] and we will use it in this paper.

To appeal to the $1/N$ expansion,[16] we scale the inverse temperature β by N. ($N\beta$ is denoted simply β or β_c in (1.1).) The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$\langle f \rangle \equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_i \delta(\phi_i^2 - N\beta)\, d\phi_i \tag{1.2}$$

Here

$$\Lambda = [-(L/2)^M, (L/2)^M)^\nu \subset \mathbf{Z}^\nu$$

is the large square with center at the origin, where L is chosen odd (e.g., $L=3$) and M is a large integer. Moreover $\phi(x) = (\phi(x)^{(1)},..., \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $\langle 1 \rangle = 1$. The Hamiltonian H_Λ is given by

$$H_\Lambda \equiv -\tfrac{1}{2} \sum_{|x-y|_1 = 1} \phi(x)\, \phi(y) \tag{1.3}$$

where $|x|_1 = \sum_{i=1}^{\nu} |x_i|$.

First substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]\, da/2\pi$ into Eq. (1.2) with the condition that $\operatorname{Im} a_i < -\nu$.[1, 2] We set

$$\operatorname{Im} a_i = -\left(\nu + \frac{m^2}{2}\right), \qquad \operatorname{Re} a_i = \frac{1}{\sqrt{N}} \psi_i \tag{1.4}$$

where $m > 0$ will be determined soon. Thus we have

$$
\begin{aligned}
Z_\Lambda &= c^{|\Lambda|} \int \cdots \int \exp\left[-\frac{1}{2} \left\langle \phi, \left(m^2 - \Delta + \frac{2i}{\sqrt{N}} \psi \right) \phi \right\rangle \right. \\
&\quad \left. + \sum_j i \sqrt{N}\, \beta \psi_j \right] \prod \frac{d\phi_j\, d\psi_j}{2\pi} \\
&= c^{|\Lambda|} \det(m^2 - \Delta)^{-N/2} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi}
\end{aligned} \tag{1.5}
$$

where c's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian and

$$
F(\psi) = \det\left(1 + \frac{2iG}{\sqrt{N}} \psi \right)^{-N/2} \exp\left[i \sqrt{N}\, \beta \sum_j \psi_j \right] \tag{1.6}
$$

Moreover $G = (m^2 - \Delta)^{-1}$ is the covariant matrix discussed later. In the same way, the two-point function is given by

$$
\langle \phi_0 \phi_x \rangle = \frac{1}{\tilde{Z}} \int \cdots \int \left(m^2 - \Delta + \frac{2i}{\sqrt{N}} \psi \right)^{-1}_{0x} F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{1.7}
$$

where the constant $\tilde{Z}$ is chosen so that $\langle \phi_0^2 \rangle = N\beta$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where

$$
G(x) = \int \frac{e^{ipx}}{m^2 + 2\sum (1 - \cos p_i)} \prod_{i=1}^{\nu} \frac{dp_i}{2\pi} \tag{1.8}
$$

This is possible for any β if $\nu \leqslant 2$, and we easily find that

$$
m^2 \sim 32 e^{-4\pi\beta} \qquad \text{for} \quad \nu = 2 \tag{1.9}
$$

as $\beta \to \infty$, which is consistent with the renormalizaiton group analysis, see e.g., ref. 5 and references cited therein:

$$
m^2 = 32 \left(\frac{e^{1-\pi/2}}{8} \right)^{\frac{-2}{N-2}} \Gamma\left(1 + \frac{1}{N-2} \right) \left[\frac{2\pi\beta}{1 - 2/N} \right]^{\frac{2}{N-2}} \exp\left[-\frac{4\pi\beta}{1 - 2/N} \right] \tag{1.10}
$$

Thus for $\nu = 2$, we can rewrite

$$
F(\psi) = \det{}_3^{-N/2} \left(1 + \frac{2iG}{\sqrt{N}} \psi \right) \exp[-\mathrm{Tr}(G\psi)^2] \tag{1.11}
$$

$$
\det{}_3(1 + A) \equiv \det[(1 + A)\, e^{-A + A^2/2}] \tag{1.12}
$$

and the determinant $\det_3(1+2iG\psi/\sqrt{N})^{-N/2}$ is regarded as a small perturbation to the Gaussian measure $\sim \exp[-\mathrm{Tr}(G\psi)^2] \prod d\psi$. Namely $F(\psi) = \det_2^{-N/2}(1+2iG\psi/\sqrt{N})$ looks like $\det^{-N/4}(1+4G\psi G\psi/N)$ which is strictly positive. If $F(\psi) \geqslant 0$ is justified, then from Eq. (1.7), we have

$$\langle \phi_0 \phi_x \rangle = \frac{1}{\tilde{Z}} \int \cdots \int \left(m^2 - \Delta + \frac{2i}{\sqrt{N}} \psi \right)^{-1}_{0x} F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{1.13}$$

$$\leqslant \left| \sup_{\psi} \left(m^2 - \Delta + \frac{2i}{\sqrt{N}} \psi \right)^{-1}_{0x} \right|$$

$$\leqslant (m^2 - \Delta)^{-1}_{0x} \leqslant c_1 \exp(-c_2 m\,|x|) \tag{1.14}$$

where $c_i > 0$ are suitable constants. Namely the (approximate) positivity of the subtracted determinant $F(\psi)$ ensures exponential clustering of the correlation functions.

This argument fails, however, for large β since the expansion of the determinant cannot be justified anymore if $|\psi| > e^{-4\pi\beta} N^{1/2}$ (note that $\|G\| = m^{-2} \sim e^{4\pi\beta}$) and we cannot expect the approximate positivity of the subtracted determinant $F(\psi)$. But this argument still works if the main contribution of the ψ integral comes from small ψ region (say $|\psi| \ll \sqrt{N}\,\beta^{-1/2}$ since $G(x, y) \sim \beta$ for $|x-y| < m^{-1}$). If so, we can expand the determinant, and the approximate positivity of $F(\psi)$ and exponential clustering of the correlation functions remain to hold.

This scenario, of course, crucially depends on properties of the measure

$$dv = \text{const.}\ F(\psi) \prod d\psi(x) \tag{1.15}$$

To study the properties of $F(\psi)$, we decompose the determinant into a product of many determinants which are expandable and easy to analyze. The block spin transformation is most convenient for this purpose, and we can confirm our conjecture by applying the block spin transformation to the system.

The most important point in this approach is that we must regard the auxiliary field ψ as a marginal field operator since $[G^{\circ 2}]^{-1}(x, y) \sim |x-y|^{-4}$ where $G^{\circ 2}(x, y) = G(x, y)^2$ so that $\mathrm{Tr}(G\psi)^2 = \langle \psi, G^{\circ 2}\psi \rangle$. This justifies that we can neglect the subtract determinants at almost all levels of the block spin transformations. A technically important theorem is Theorem 4 in Section 3 which gives $\int \psi(x)\, \psi(y)\, dv$ ($\sim [G^{\circ 2}](x, y)$) through the multiscale decompositions (block-spin transformations) of ψ. This theorem shows that the two-point correlation function of the fluctuation field $\tilde{\psi}_n$ of ψ at the distance scale L^n is given approximately by $(\beta_{n+1}\Gamma_n)^{-1}$, where Γ_n is the covariant matrix (propagator) of the fluctuation field ξ_n of the block spin ϕ_n and β_{n+1} is the effective inverse temperature at the distance scale L^{n+1}. The following is a non-rigorous summary of our results from technical point of view:

Theorem. The auxiliary field ψ is a marginal field operator. As a result, the effect of $\det_3^{-N/2}(1+2iG\psi/\sqrt{N})$ in $F(\psi)$ is small no matter how large β is, namely

$$dv \sim \text{const. } \exp[-\langle \psi, G^{\circ 2}\psi \rangle] \prod d\psi(x) \tag{1.16}$$

Moreover the two-point function $\int \psi(x)\, \psi(y)\, dv$ is represented by a linear combination of the covariances of the fluctuation fields $\tilde{\psi}_n$, and

$$\int \psi(x)\, \psi(y)\, dv \sim \text{const.} \frac{1}{\beta(x, y)\, |x-y|^4} \sim \frac{1}{G^{\circ 2}}(x, y) \tag{1.17}$$

for $|x-y| < m^{-1}$ where $\beta(x, y) \sim \beta - (2\pi)^{-1} \log(|x-y|)$.

This paper is the first part of the realization of this scenario: we extract the main part of the the block spin transformation and show that the renormalization group flow moves along our scenario.

We organize the paper as follows: In Section 2, we introduce our block spin transformation applied to the system and obtain the first and the second recursion relations. Section 3 is the most technical part in this paper: we obtain the recursion formulas for general n through our simplifications which will be justified, and we solve it. In Section 4, we obtain the effective interaction V_n and the inverse temperature β_n at the distance scale L^n. Some conclusions are given in the final Section 5. In the Appendices, we establish some properties of the Green functions used in the paper.

2. DERIVATION OF THE BST OF THE $O(N)$ SPIN MODEL

2.1. The BST of the $O(N)$ Spin Model

To realize our scenario sketched in the introduction, we decompose the determinant into product of determinants each of which comes from the integration over the fluctuation field of ϕ. Namely we decompose $\Lambda = [-(L/2)^M, (L/2)^M)^2 \subset Z^2$ into blocks (squares) $\square_i$ of size $L \times L$ (with L around 3 or 4), and repeat the following steps:

(1) integrate over the fluctuation fields $\xi(x)$

(2) integrate over $\psi(x)$ keeping the block sums fixed,

(3) collect the resultant terms so that the original form would be recovered with small change of coefficients.

If there are blocks $\square_i$ where ψ takes a large value which prohibits the expansion of the determinant, we will use an a priori estimate. The small field region is a collection of squares on which the expansion of the determinant converges absolutely.

We now set

$$W_0(\phi, \psi) = \frac{1}{2} \langle \phi, G_0^{-1} \phi \rangle - i \sum_x J_0(x)\, \psi(x) \tag{2.1}$$

$$J_0 = \sqrt{N}\, \beta - \frac{1}{\sqrt{N}} \phi^2(x) = -\frac{1}{\sqrt{N}} :\phi^2(x):_{G_0} \tag{2.2}$$

$$G_0^{-1} = -\Delta + m^2 \tag{2.3}$$

where $:A:_{G_0}$ is the Wick product of A with respect to the Gaussian measure of $d\mu_{G_0}$ with mean zero and covariance G_0.

For the reader who may not be familiar with the block spin transformations formulated in a mathematically rigorous way, we just sketch how to define block spins and fluctuation fields so that they are independent. We want to represent original spins $\phi(x) \equiv \phi_0(x)$ and $\psi(x) \equiv \psi_0(x)$ by the block spin variables $\phi_1(x) = (C\phi)(x)$ and $\psi_1(x) = (C'\psi)(x)$, and by the fluctuation fields $\xi(\zeta)$ and $\tilde{\psi}(\zeta)$, $\zeta \in \Lambda - L\Lambda_1$ where

$$(C\phi)(x) \equiv L^{-2} \sum_{\zeta \in \square} \phi(Lx + \zeta) \tag{2.4}$$

$$(C'\psi)(x) \equiv L^2 (C\psi)(x) = \sum_{\zeta \in \square} \psi(Lx + \zeta) \tag{2.5}$$

$$\Lambda_n = Z^2 \cap L^{-n} \Lambda, \qquad n = 1, 2, \ldots \tag{2.6}$$

and $\square$ is the box of size $L \times L$ center at the origin. The operator C takes the arithmetic averages of ϕ over the blocks, the operator C' takes sum of ψ over the blocks, and the both subsequently reduce the coordinates by $1/L$. We do not introduce any scaling factor in the definition of the block spins $\psi_{n+1}(x) = \sum_{\zeta \in \square} \psi_n(Lx + \zeta)$. The reason for this is that the auxiliary field $\{\psi(x)\}$ interacts anti-ferromagnetically (ψ appears as $\exp[-\mathrm{Tr}(G\psi)^2]$).

Remark 1. In general, C and C' are chosen so that G and $[G^{\circ 2}]^{-1}$ are left invariant by C and C', respectively:

$$(CGC^{+})(x, y) \sim G(x, y), \qquad (C'[G^{\circ 2}]^{-1}\, C'^{+})(x, y) \sim [G^{\circ 2}]^{-1}\,(x, y) \tag{2.7}$$

It is easy to check that our C and C' satisfy these requirements since $G(x, y) \sim \beta-(2\pi)^{-1}\log(|x-y|)$ and $[G^{\circ 2}]^{-1}\,(x, y) \sim |x-y|^{-4}$. (We can show $[G^{\circ 2}]^{-1}\,(x, y) \sim \beta^{-1}\,|x-y|^{-4}$ by the Fourier transformation, see Appendix.)

The fluctuation fields $\xi_0(\zeta)$ and $\tilde{\psi}(\zeta)$ are living on $\Lambda - L\Lambda_1$ and the block spin $\phi_1(x)$ has the covariance

$$G_1(x, y) = CGC^{+}(x, y) = L^{-4} \sum_{\zeta_1, \zeta_2 \in \square} G_0(Lx+\zeta_1, Ly+\zeta_2) \tag{2.8}$$

where $G_0 = G$. Similarly we define

$$G_n(x, y) = CG_{n-1}C^{+}(x, y) = L^{-4} \sum_{\zeta_1, \zeta_2 \in \square} G_{n-1}(Lx+\zeta_1, Ly+\zeta_2) \tag{2.9}$$

$$A_n(x, y) = G_{n-1}C^{+}G_n^{-1}(x, y) = \sum_{\zeta} G_{n-1}(x, \zeta)[C^{+}G_n^{-1}](\zeta, y) \tag{2.10}$$

We introduce a map Q: $R^{\Lambda\backslash L\Lambda_1} \to R^{\Lambda}$ and its adjoint Q^{+}: $R^{\Lambda} \to R^{\Lambda\backslash L\Lambda_1}$:

$$(Q\xi)(x) = \begin{cases} \xi(x) & \text{if} \quad x \notin LZ^2 \\ -\displaystyle\sum_{y \in \square(x)} \xi(y) & \text{if} \quad x \in LZ^2 \end{cases} \tag{2.11}$$

$$(Q^{+}f)(x) = f(x) - f(x_0), \qquad x \in \Lambda\backslash L\Lambda_1 \tag{2.12}$$

where $x_0 \in L\Lambda_1$ is the nearest point to x. Namely Q^{+} acts as a differentiation.

The substitution $\phi_n(x) = A_{n+1}\phi_{n+1} + Q\xi_n$ $(n = 0, 1,...)$ decomposes the Gaussian measure $d\mu_{G_n}$ into the product of two Gaussian measures $d\mu_{G_{n+1}}$ and $d\mu_{\Gamma_n}$:

$$d\mu_{G_n}(\phi_n) = d\mu_{G_{n+1}}(\phi_{n+1})\, d\mu_{\Gamma_n}(\xi_n) \tag{2.13}$$

since

$$\langle \phi_n, G_n^{-1}\phi_n \rangle = \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} \rangle + \langle \xi_n, \Gamma_n^{-1}\xi_n \rangle \tag{2.14}$$

$$\Gamma_n^{-1} = Q^{+}G_n^{-1}Q \tag{2.15}$$

The point is that Γ_n decays exponentially fast uniformly in n ($c' = O(1/L)$ is a positive constant):

$$|\Gamma_n(x, y)| \leqslant O(1)\, e^{-c'\,|x-y|}, \qquad x, y \in \Lambda_n \setminus L\Lambda_{n+1} \tag{2.16}$$

Thus we have the recursion relations

$$\varphi_n(x) = \varphi_{n+1}(x) + z_n, \qquad n = 0, 1,... \tag{2.17}$$

where

$$\varphi_n = \mathscr{A}_n \phi_n, \qquad z_n = \mathscr{A}_n Q \xi_n, \qquad \mathscr{A}_n = A_1 A_2 \cdots A_n = G_0 (C^+)^n\, G_n^{-1} \tag{2.18}$$

and we put $\mathscr{A}_0 = A_0 = 1$. Therefore

$$\frac{1}{N} \int \varphi_n(x)\, \varphi_n(y)\, d\mu_{G_n} \equiv \mathscr{G}_n(x, y) = \mathscr{A}_n G_n \mathscr{A}_n^+(x, y) \tag{2.19}$$

$$\frac{1}{N} \int z_n(x)\, z_y(y)\, d\mu_{G_n} \equiv \mathscr{T}_n(x, y) = \mathscr{A}_n Q \Gamma_n Q^+ \mathscr{A}_n^+(x, y) \tag{2.20}$$

where $x, y \in \Lambda$. We list their properties (see ref. 7 or Appendix for the proof):

Lemma 1. The following bounds hold:

$$|\mathscr{A}_n(\zeta, x)| \leqslant O(1) \exp\left[-c \left|\frac{\zeta}{L^n} - x\right|\right] \tag{2.21a}$$

$$|\mathscr{A}_n(\zeta, x) - \mathscr{A}_n(\xi, x)| \leqslant O(1) \frac{|\zeta - \xi|}{L^n} \left(\exp\left[-c\left|\frac{\zeta}{L^n} - x\right|\right] + \exp\left[-c\left|\frac{\xi}{L^n} - x\right|\right]\right) \tag{2.21b}$$

$$|\mathscr{T}_n(x, y)| \leqslant O(1) \exp\left[-\frac{c}{L^{n+1}}\, |x-y|\right] \tag{2.21c}$$

$$|\mathscr{T}_n(x, y) - \mathscr{T}_n(x', y)| \leqslant O(1) \frac{|x - x'|}{L^n} \left(\exp\left[-\frac{c}{L^{n+1}}\, |x-y|\right] + \exp\left[-\frac{c}{L^{n+1}}\, |x'-y|\right]\right) \tag{2.21d}$$

where $\zeta, \xi \in \Lambda$, $x, y \in \Lambda_n$, in the first two equations and $x, x', y \in \Lambda$ in the last two equations. The positive constants c's are chosen independent of L.

2.2. The First Block Spin Transformation

What kind of the decompositions $\psi_n = \tilde{A}_{n+1}\psi_{n+1} + Q\tilde{\psi}_n$ is most appropriate for the auxiliary field ψ? We will see that $\{\tilde{A}_{n+1}\psi_{n+1}, \tilde{\psi}_n\}$ are inevitably chosen to (approximately) factorize the Gaussian measure

$$\exp[-\mathrm{Tr}(G\psi)^2] \prod d\psi = \exp[-\langle \psi, G^{\circ 2}\psi\rangle] \prod d\psi \tag{2.22}$$

as in the case of $d\mu_G$. Here for given matrices A and B, the Hadamard product of A and B is given by $(A \circ B)(x, y) \equiv A(x, y)\, B(x, y)$ and we put $A^{\circ 2} = A \circ A$. In fact $\mathscr{G}_0^{\circ 2} = \mathscr{G}_1^{\circ 2} + (\mathscr{G}_0^{\circ 2} - \mathscr{G}_1^{\circ 2})$, $(G = \mathscr{G}_0)$, and we will see that $\mathscr{G}_0^{\circ 2} - \mathscr{G}_1^{\circ 2}$ is obtained as the Hamiltonian of ψ by the integrations over ξ_0. Thus the integrations by the fluctuations ξ_n yield a natural multiscale decomposition $G^{\circ 2} = \sum_n (\mathscr{G}_n^{\circ 2} - \mathscr{G}_{n+1}^{\circ 2})$.

Let us see explicitly how this program works for $n = 1$. Substitute $\phi_0(x) = (A_1\phi_1)(x) + (Q\xi_0)(x)$ and $\psi(x) = (\tilde{A}_1\psi_1)(x) + Q\tilde{\psi}(x)$ into $F_0(\phi_0, \psi_0) \equiv \exp[-W_0(\phi_0, \psi_0)]$, and integrate over $\{\xi_0(x), \tilde{\psi}_0(x); x \in \Lambda \backslash L\Lambda_1\}$:

$$F_1(\phi_1, \psi_1) = \int F_0(A_1\phi_1 + Q\xi_0, \tilde{A}_1\psi_1 + Q\tilde{\psi}) \prod d\xi_0(x) \prod d\tilde{\psi}(x) \tag{2.23}$$

where $\{\tilde{\psi}_x; x \in \Lambda \backslash L\Lambda_1\}$ is the fluctuation field of ψ_0 and the matrix $\tilde{A}_1$ will be determined later. The integral of (2.23) over ξ_0 is carried out and we have

$$\begin{aligned}
\int \exp\Bigg[&-\frac{1}{2}\{\langle \phi_1, G_1^{-1}\phi_1\rangle + \langle \xi_0, Q^+ G_0^{-1} Q\xi_0\rangle\} - \frac{2i}{\sqrt{N}} \sum_x j_0(x)\, \xi_0(x) \\
&+ i\sqrt{N} \sum_x \left(\beta - \frac{1}{N}[\varphi_1(x)^2 + (Q\xi_0)_x^2]\right)\psi(x)\Bigg] \prod d\xi_0(x) \\
= \det{}_3^{-N/2} &\left(1 + \frac{2i}{\sqrt{N}}\Gamma_0 Q^+\psi Q\right) \\
&\times \exp\Bigg[-\frac{1}{2}\langle \phi_1, G_1^{-1}\phi_1\rangle + i\sqrt{N}\sum_x \left(\beta - Q\Gamma_0 Q^+ - \frac{1}{N}\varphi_1^2(x)\right)\psi(x) \\
&- \left\langle \psi, \left[(Q\Gamma_0 Q^+)^{\circ 2} + \frac{2}{N}\left(Q\frac{1}{P}Q^+\right) \circ (\varphi_1\varphi_1)\right]\psi\right\rangle\Bigg]
\end{aligned} \tag{2.24}$$

except for the trivial constant, where $\varphi_1(x)=(A_1\phi_1)(x)$, $(x\in\Lambda)$, $\Gamma_0=[Q^+(-\Delta+m^2)\,Q]^{-1}$ and

$$P\equiv Q^+\left(-\Delta+m^2+\frac{2i}{\sqrt{N}}\psi\right)Q=\Gamma_0^{-1}+\frac{2i}{\sqrt{N}}Q^+\psi Q \tag{2.25}$$

$$j_0(x)=\sum_\zeta \varphi_1(\zeta)\,Q(\zeta,x)\,\psi(\zeta)=[Q^+(\varphi_1\cdot\psi)](x) \tag{2.26}$$

Both Γ_0 and P^{-1} are maps from $R^{\Lambda\backslash L\Lambda_1}$ into itself. Note that P^{-1} exhibits uniform exponential decay ($P^{-1}\sim\Gamma_0$) and that no matter how small m^2 is, Γ_1 has the mass of order $(m^2+L^{-2})^{1/2}$, and the determinant has uniform locality.

We define the small field of $\psi(x)$ by

$$\left|\frac{2}{\sqrt{N}}\Gamma_0 Q^+\psi Q\right|<N^{-\varepsilon} \tag{2.27}$$

uniformly in $\beta>0$. Since Γ_0 is a bounded operator, this requirement is equivalent to

$$|(Q^+\psi Q)_{ij}|=|\psi_i\delta_{ij}+\psi_x\delta_{[i/L],[j/L]}|<N^\delta,\qquad \delta<\tfrac{1}{2} \tag{2.28}$$

where $i,j\in\Lambda\backslash L\Lambda_1$, $[i/L]$ stands for the point $\in Z^2$ nearest to $(i_1/L, i_2/L)$ and x is the point $\in L\Lambda_1$ nearest to $i=(i_1,i_2)$. Therefore

$$|(Q^+\psi Q)_{ij}|<N^\delta \text{ if and only if } |\psi(x)|<N^\delta \text{ for all } x \text{ in the block} \tag{2.29}$$

But this constraint does not seem to play an important role if β is large (we need $N\geqslant 3$ for the integrability of the determinant) since $|\psi(x)|<$ const. $\beta^{-1/2}$ is required for the factor $\exp[-\langle\psi, G^{\circ 2}\psi\rangle]$ small. This seems to a benefit of the block-spin transformation, see refs. 14 and 15.

In the small field region, we may neglect $\det_3^{-N/2}(\cdots)$ and we consider the effective interaction term of ψ given in the final exponential of Eq. (2.24):

$$\exp\left[-\tfrac{1}{2}\langle\phi_1, G_1^{-1}\phi_1\rangle-\langle\psi,\hat{H}_0^{-1}\psi\rangle+i\sum J_1(x)\,\psi(x)\right] \tag{2.30}$$

where

$$\hat{H}_0^{-1}=(Q\Gamma_0Q^+)^{\circ 2}+\frac{2}{N}\left[\left(Q\frac{1}{P}Q^+\right)\circ(\varphi_1\cdot\varphi_1)\right] \tag{2.31}$$

$$
\begin{aligned}
J_1(x) &= \sqrt{N}\,\beta - \sqrt{N}\,(Q\Gamma_0 Q^+)(x,x) - \frac{1}{\sqrt{N}}\,(\varphi_1)_x^2 \\
&= \sqrt{N}\,(G_0 C^+ G_1^{-1} C G_0)(x,x) - \frac{1}{\sqrt{N}}\,(G_0 C^+ G_1^{-1}\phi_1)_x^2 \\
&= -\frac{1}{\sqrt{N}}\,{:}\varphi_1^2(x){:}_{G_1}
\end{aligned}
\tag{2.32}
$$

and $:\,:_{G_1}$ denotes the Wick product with respect to the Gaussian measure $d\mu_{G_1}$,

$$
{:}\varphi_i(x)\,\varphi_j(y){:}_{G_1} \equiv \varphi_i(x)\,\varphi_j(y) - \delta_{ij}(A_1 G_1 A_1^+)(x,y)
$$

$$
{:}\varphi_1\cdot\varphi_1{:}_{G_1}\,(x,y) \equiv \sum_i\,{:}\varphi_{1,i}(x)\,\varphi_{1,i}(y){:}_{G_1}
$$

and we have used $Q\Gamma_0 Q^+ = G_0 - G_0 C^+ G_1^{-1} C G_0$ and $A_1 = G_0 C^+ G_1^{-1}$.

We extract the main part $\tilde{H}_0^{-1}$ from $\hat{H}_0^{-1}$ by replacing $\varphi_1(x)\,\varphi_1(y)$ by $N\mathscr{G}_1(x,y)$:

$$
\hat{H}_0^{-1} \equiv \tilde{H}_0^{-1} + \delta H_0^{-1} \tag{2.33}
$$

$$
\tilde{H}_0^{-1} \equiv \mathscr{T}_0^{\circ 2} + 2\mathscr{T}_0 \circ \mathscr{G}_1 = \mathscr{G}_0^{\circ 2} - \mathscr{G}_1^{\circ 2} \tag{2.34}
$$

$$
\delta H_0^{-1} = \frac{2}{N}\,[\mathscr{T}_0 \circ {:}\varphi_1\cdot\varphi_1{:}_{G_1}] + \frac{2}{N}\left[\left(Q\left(\frac{1}{P} - \Gamma_0\right)Q^+\right)\circ(\varphi_1\cdot\varphi_1)\right] \tag{2.35}
$$

where ${:}\varphi_1(x)\,\varphi_1(y){:}_{G_1} \equiv \varphi_1(x)\,\varphi_1(y) - N\mathscr{G}_1(x,y)$ and we have set

$$
\mathscr{T}_0 = Q\Gamma_0 Q^+, \qquad \mathscr{G}_0 = A_0^+ G_0 A_0 = G_0, \qquad \mathscr{G}_1 = A_1^+ G_1 A_1 \tag{2.36}
$$

so that $\mathscr{T}_0 = \mathscr{G}_0 - \mathscr{G}_1$. The relation (2.34) is what we have claimed in the beginning of this section. Therefore to decompose the Hamiltonian $\tilde{H}_0^{-1}$, we should put

$$
\tilde{A}_1 = \tilde{H}_0 (C')^+ H_1^{-1}, \qquad H_1 = C' \tilde{H}_0 (C')^+ \tag{2.37}
$$

To calculate $\tilde{A}_1$ and H_1, let P_0 (resp. P_1) be the projection operator onto the set of the block-wise constant functions $\{\psi_1([x/L])\}$ (resp. to the set of the zero-average functions $\{Q\xi\}$):

$$
P_0 = \frac{1}{L^2}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \qquad P_1 = 1 - P_0 \tag{2.38}
$$

Then we have

$$P_0\mathcal{T}_0P_0=0, \qquad P_1\mathcal{T}_0P_1>O(1) \tag{2.39}$$

Since $\mathcal{G}_1(x,y)\sim\beta$ is slowly varying for $|x-y|<m^{-1}$ and $\Gamma_0(x,y)$ is rapidly decreasing in $|x-y|$, we see that

$$|\mathcal{T}_0(x,y)[\mathcal{G}_1(x,y)-\beta_1]|\leqslant \text{const.}\exp(-c\,|x-y|) \tag{2.40}$$

where $\beta_1\equiv\mathcal{G}_0(Lx,Lx)$ and $c=O(1/L)>0$ uniformly in β. Therefore

$$\begin{aligned}(\mathcal{T}_0\circ\mathcal{G}_1)(x,y)&=\beta_1\mathcal{T}_0(x,y)+\mathcal{T}_0(x,y)[\mathcal{G}_1(x,y)-\beta_1]\\&\sim\beta_1\mathcal{T}_0(x,y)\end{aligned} \tag{2.41}$$

Put $M\equiv M_0=\tilde{H}_0^{-1}$, $M_{ij}=P_iMP_j$ and $\tilde{H}_{ij}=P_i\tilde{H}_0P_j$ so that $M_0=\{M_{ij}\}$ and $\tilde{H}_0=\{\tilde{H}_{ij}\}$. Then we see that

$$\begin{aligned}M_{00}&\equiv P_0MP_0\sim P_0\mathcal{T}_0^{\circ 2}\\M_{11}&\equiv P_1MP_1\sim 2\beta_1P_1\mathcal{T}_0P_1=O(\beta_1)\end{aligned} \tag{2.42}$$

are strictly positive operators of short range. To see that M_{10} is $O(1)$ uniformly in β, we remark that M_{10} is given by

$$M_{10}=P_1[\mathcal{T}_0^{\circ 2}+2\mathcal{T}_0\circ(\mathcal{G}_1-\beta_1)]\,P_0 \tag{2.43}$$

since $\beta_1\mathcal{T}_0P_0=0$, where $(\mathcal{T}_0\circ(\mathcal{G}_1-\beta_1))(x,y)$ is bounded by $\exp(-c\,|x-y])$, $c=O(1/L)$ uniformly in β.

Since $\tilde{H}_0=\{\tilde{H}_{ij}\}$ is given by

$$\tilde{H}_{00}=[M_{00}-M_{01}M_{11}^{-1}M_{10}]^{-1}, \qquad \tilde{H}_{01}=-M_{00}^{-1}M_{01}\tilde{H}_{11} \tag{2.44}$$

$$\tilde{H}_{11}=[M_{11}-M_{10}M_{00}^{-1}M_{01}]^{-1}, \qquad \tilde{H}_{10}=-M_{11}^{-1}M_{10}\tilde{H}_{00} \tag{2.45}$$

we have (see Appendix B for more precise arguments)

$$\tilde{A}_1=\tilde{H}_0(C')^+\,[C'\tilde{H}_0(C')^+]^{-1}=C^+-M_{11}^{-1}M_{10}C^+ \tag{2.46}$$

$$H_1=C'\tilde{H}_{00}C'^+=L^4C[M_{00}-M_{01}M_{11}^{-1}M_{10}]^{-1}\,C^+ \tag{2.47}$$

where C is the block spin operator defined before:

$$C(x,y)=\frac{1}{L^2}\,\delta_{x,[y/L]}, \qquad C^+(x,y)=\frac{1}{L^2}\,\delta_{[x/L],y}$$

We can calculate $-M_{11}^{-1}M_{10}C^{+}$ in $\tilde{A}_1$ easily. In fact $M_{11}(x, y) = (1-P_0)\, M(1-P_0) = Q^{+}MQ$ on $QR^{\Lambda\backslash L\Lambda_1}$, and then we have

$$M_{11}^{-1}(x, y) = (Q^{+}MQ)^{-1} \sim (2\beta_1 Q^{+}Q\Gamma_0 Q^{+}Q)^{-1} \tag{2.48}$$

$$= \frac{1}{2\beta_1}(Q^{+}Q)^{-1}\,[Q^{+}(-\Delta+m^2)\,Q](Q^{+}Q)^{-1} \tag{2.49}$$

on $QR^{\Lambda\backslash L\Lambda_1}$, where $(Q^{+}Q)^{-1}$: $R^{\Lambda\backslash L\Lambda_1} \to R^{\Lambda\backslash L\Lambda_1}$ is given by

$$(Q^{+}Q)^{-1} = 1 - \mathscr{P} + \frac{1}{L^2}\mathscr{P}$$

with $\mathscr{P}$ being the projection onto the set of the block-wise $(\square_{Lx}\backslash\{Lx\})$ constant functions. To rewrite this as the map from $QR^{\Lambda\backslash L\Lambda_1}$ into $QR^{\Lambda\backslash L\Lambda_1}$, we multiply Q to the left and Q^{+} to the right. Finally we note that $\sum_y \mathscr{G}_n(x, y)$ are independent of x, which means that $\sum_y M(x, y)$ is independent of x. Therefore $\sum_{y\in\Lambda_1}(M_{10}C^{+})(x, y) = 0$ and we obtain

Lemma 2. For large $\beta \gg L$, $\tilde{A}_1$ is almost diagonal and has no tail, that is

$$\tilde{A}_1(x, y) = \frac{1}{L^2}\delta_{[\frac{x}{L}],\, y} + \frac{1}{\beta_1}\delta\tilde{A}_1(x, y), \qquad x \in \Lambda, \quad y \in \Lambda_1 \tag{2.50a}$$

$$\delta\tilde{A}_1(x, y) = O(e^{-O(1)\,|x/L-y|}) \tag{2.50b}$$

where

$$\sum_{\zeta\in\square}\delta\tilde{A}_1(Lx_1+\zeta, y) = 0, \qquad \sum_{y\in\Lambda_1}\delta\tilde{A}_1(x, y) = 0 \tag{2.51}$$

Moreover $\tilde{H}_1$ is a strictly positive operator of order $O(1)$ of short range:

$$H_1^{-1}(x, y) = \frac{1}{L^4}\sum_{\zeta,\,\xi\in\square} M_0^{-1}(Lx+\zeta, Ly+\xi)$$

$$\sim \frac{1}{L^4}\sum_{\zeta,\,\xi\in\square}(Q\Gamma_0 Q^{+})^{\circ 2}\,(Lx+\zeta, Ly+\xi) \sim \delta_{xy}, \qquad x, y \in \Lambda_1$$

Remark 2. Thus both $A_1(x, y)$ and $\tilde{A}_1(x, y)$ decrease exponentially fast. This is the case for all n. Then one may put

$$\phi_n(x) \sim \phi_{n+1}\left(\left[\frac{x}{L}\right]\right) + (Q\xi_n)(x)$$

$$\psi_n(x) \sim \frac{1}{L^2}\psi_{n+1}\left(\left[\frac{x}{L}\right]\right) + (Q\tilde{\psi}_n)(x)$$

Thus $\phi_n^p(x)$, $\psi_n^p(x)$ $(p > 1)$ and $\phi_n^2(x)\,\psi_n(x)$ are relevant, irrelevant and marginal, respectively.

Therefore the following set $\mathscr{K}_1$ of the smooth-small field seems to dominate the functional integral (the condition (2.52a) may be stronger than necessary, see ref. 13. $\mathscr{K}_n$ will be defined in the same way):

Definition 1. The smooth-small field $\mathscr{K}_1(X)$ is the subset of $\{\phi_1(x), \psi_1(x); x \in \Lambda_1\}$ such that

$$||\varphi_1(x)| - \sqrt{N\mathscr{G}_1(x, x)}| < \frac{N^\varepsilon}{\sqrt{\beta}} \tag{2.52a}$$

$$|\varphi_1(x + e_\mu) - \varphi_1(x)| < N^{1/2+\varepsilon} \tag{2.52b}$$

$$|\psi_1(x)| < N^\alpha \tag{2.52c}$$

$$|(\tilde{A}_1\psi_1)(x + e_\mu) - (\tilde{A}_1\psi_1)(x)| < \frac{N^\varepsilon}{\sqrt{\beta}} \tag{2.52d}$$

for all $x \in X$, where $0 < \alpha < 1/2$ and $0 < \varepsilon < \alpha$ are small positive constants.

Remark 3. Note that the constraint (2.52a) means that

$$|{:}\phi_1^2(x){:}| = (|\phi_1(x)| + \sqrt{N\beta}|)\,||\phi_1(x)| - \sqrt{N\beta}| \leqslant \text{const. } N^{1/2+\varepsilon} \tag{2.53}$$

This also means that the fluctuation parallel with ϕ_1 is very small, which is the feature seen in the hierarchical model of Dyson–Wilson type with large N. Moreover the definition of $\mathscr{K}_1$ means that $P_0 \delta H_0^{-1} P_0$ and $P_1 \delta H_0^{-1} P_1$ are sufficiently small on $\mathscr{K}_1$.

We now have to integrate over ψ. We note that ψ is again Gaussian random variable of mean zero and covariance $\frac{1}{2}(H_0^{-1} + \delta H_0^{-1})^{-1}$. But δH_0^{-1}

is small compared with H_0^{-1} on the both subsets of ψ . Then we treat δH_0^{-1} by perturbation choosing N large. Since

$$\langle\psi, \hat{H}_0^{-1}\psi\rangle = \langle\psi, \tilde{H}_0^{-1}\psi\rangle + \langle\psi, \delta H_0^{-1}\psi\rangle \tag{2.54}$$

$$\begin{aligned}\langle\psi, \tilde{H}_0^{-1}\psi\rangle + i\sum J_0(x)\,\psi(x) &= \langle\psi_1, H_1^{-1}\psi_1\rangle + i\langle J_1, \tilde{A}_1\psi_1\rangle\\ &\quad + \langle\tilde{\psi}, Q^+\tilde{H}_0^{-1}Q\tilde{\psi}\rangle + i\langle Q^+J_1, \tilde{\psi}\rangle\end{aligned} \tag{2.55}$$

if $\phi_1 \in \mathscr{K}_1$, we can integrate over $\tilde{\psi}$ and obtain the new factor

$$\det{}^{-1/2}(Q^+\tilde{H}_0^{-1}Q)\,\exp[-\mathscr{F}_1]$$

where $\mathscr{F}_1$ is a small quantity (per unit volume on $\mathscr{K}_1$) given by

$$\mathscr{F}_1 \equiv \sum_{x,\,y} f_1(x, y)(Q^+ :\varphi_1^2:_{G_1})(x)(Q^+ :\varphi_1^2:_{G_1})(y) \tag{2.56}$$

$$f_1(x, y) \equiv \frac{1}{4N}\,[Q^+\tilde{H}_0^{-1}Q]_{xy}^{-1} \tag{2.57}$$

Thus we obtain

$$\begin{aligned}\exp[-W_1(\phi_1, \psi_1)] = \exp[&-\tfrac{1}{2}\langle\phi_1, G_1^{-1}\phi_1\rangle - \langle\psi_1, H_1^{-1}\psi_1\rangle\\ &+ i\langle J_1, \tilde{A}_1\psi_1\rangle - \mathscr{F}_1 + \delta W_1]\end{aligned} \tag{2.58}$$

where δW_1 is the remainder (i.e., remaining part of the determinant and so on). This should be compared with the 0th order integrand $\exp[-W_0]$, see Eq. (2.1). Thus the approximate flow is represented by

$$J_0 = -\frac{1}{\sqrt{N}} :\phi_0^2(x):_{G_0} \to J_1 = -\frac{1}{\sqrt{N}} :\varphi_1^2(x):_{G_1} \tag{2.59}$$

$$H_0^{-1} = 0 \to H_1^{-1} = (C'\tilde{H}_0(C')^+)^{-1} \tag{2.60}$$

or simply by the conventional flow of β_k : $\beta_0 = \beta \to \beta_1 = \beta - \mathscr{T}_0(x, x)$.

2.3. The Second Block Spin Transformation

Assuming again that we are on the small smooth field $\mathscr{K}_2$, we can repeat the integrations over ξ_1 and $\tilde{\psi}_1$. The positive polynomial $\mathscr{F}_1$ may be regarded as a fraction of $\lambda(:\varphi_0^2:)^2$, ($\lambda \to \infty$). In this paper, we neglect $\mathscr{F}_1$

which yields a small renormalization effect of order $O(N^{-1})$.[13] In this case, we just repeat the previous calculations: we put

$$\phi_1 = A_2\phi_2 + Q\xi_1 \tag{2.61}$$

$$\varphi_1 = \varphi_2(x) + z_1 \equiv \mathscr{A}_2\phi_2 + \mathscr{A}_1 Q\xi_1 \tag{2.62}$$

where $A_2 = G_1 C^+ G_2^{-1}$, $\mathscr{A}_1 = A_1$ and $\mathscr{A}_2 = A_1 A_2$. Thus $\varphi_1^2(x) = \varphi_2^2(x) + 2\varphi_2(x)\, z_1(x) + z_1^2(x)$ and we have

$$\exp\left[-\tfrac{1}{2}\langle\phi_1, G_1^{-1}\phi_1\rangle\right] = \exp\left[-\tfrac{1}{2}\langle\phi_2, G_2^{-1}\phi_2\rangle - \tfrac{1}{2}\langle\xi_1, \Gamma_1^{-1}\xi_1\rangle\right] \tag{2.63}$$

Terms containing $z_1 = A_1 Q\xi_1$ are

$$\begin{aligned} &\exp\left[-\frac{1}{2}\langle\xi_1, \Gamma_1^{-1}\xi_1\rangle - \frac{i}{\sqrt{N}}\sum_x (2\varphi_2(x)\, z_1(x)\right. \\ &\qquad \left. + z_1^2(x))(\tilde{A}_1\psi_1)(x) + \delta W_1(z, \varphi)\right] \\ &\quad = \exp\left[-\frac{1}{2}\langle\xi_1, P_1\xi_1\rangle - \frac{2i}{\sqrt{N}}\sum_x j_1(x)\, \xi_1(x) + \delta W_1(z, \varphi)\right] \end{aligned} \tag{2.64}$$

where

$$P_1 = \Gamma_1^{-1} + \frac{2i}{\sqrt{N}} Q^+ A_1^+ (\tilde{A}_1\psi_1)\, A_1 Q$$

$$(Q^+ A_1^+ (\tilde{A}_1\psi_1)\, A_1 Q)_{xy} = \sum_\zeta (A_1 Q)_{x\zeta}\, (A_1 Q)_{y\zeta}\, (\tilde{A}_1\psi_1)(\zeta)$$

$$\begin{aligned} j_1(x) &= \sum_\zeta \varphi_2(\zeta)(A_1 Q)(\zeta, x)(\tilde{A}_1\psi_1)(\zeta) \\ &= [Q^+ A_1^+ (\varphi_2 \tilde{A}_1\psi_1)](x) \end{aligned}$$

and δW_1 is the collection of higher order terms.

Then the integration over ξ_1 yields the following main integrand:

$$\det{}^{-N/2}\left(1 + \frac{2i}{\sqrt{N}}\, t_1^+ (\tilde{A}_1\psi_1)\, t_1\right) \exp\left[-\frac{2}{N}\left\langle j_1, \frac{1}{P_1} j_1\right\rangle\right]$$

where $t_1 \equiv A_1 Q \Gamma_1^{1/2}$ (since $\Gamma_1 > 0$). We expand the determinant up to the second order and extract the ψ_1^2 term from $\langle j_1, P_1^{-1} j_1 \rangle$ by replacing $\varphi_2 \varphi_2$ by $N\mathscr{G}_2$:

$$\begin{aligned}\frac{2}{N}\left\langle j_1, \frac{1}{P_1} j_1 \right\rangle &= \frac{2}{N}\langle \tilde{A}_1\psi_1, [\mathscr{T}_1 \circ (\varphi_2 \cdot \varphi_2)]\, \tilde{A}_1\psi_1 \rangle + O(N^{-2}) \\ &= 2\langle \tilde{A}_1\psi_1, [\mathscr{T}_1 \circ \mathscr{G}_2]\, \tilde{A}_1\psi_1 \rangle \\ &\quad + \frac{2}{N}\langle \tilde{A}_1\psi_1, [\mathscr{T}_1 \circ :\varphi_2 \cdot \varphi_2:]\, \tilde{A}_1\psi_1 \rangle + O(N^{-2})\end{aligned}$$

where $\mathscr{T}_1 = A_1 Q \Gamma_1 Q^+ A_1^+$ and $\mathscr{G}_2 = \mathscr{A}_2 G_2 \mathscr{A}_2^+$. We thus finally integrate over $\tilde{\psi}_1$ defined by $\psi_1 = \tilde{A}_2 \psi_2 + Q\tilde{\psi}_1$. Omitting again δH_1^{-1}, we see that the final integrand is given by

$$\exp\left[i\sqrt{N} \left\langle \left(\beta - \mathscr{T}_0 - \mathscr{T}_1 - \frac{1}{N}\varphi_2^2 \right), \tilde{A}_1\psi_1 \right\rangle - \langle \psi_1, \tilde{H}_1^{-1}\psi_1 \rangle \right] \tag{2.65}$$

where, as we have claimed,

$$\tilde{H}_1^{-1} = H_1^{-1} + \tilde{A}_1^+ M_1 \tilde{A}_1, \qquad M_1 = \mathscr{T}_1^{\circ 2} + 2\mathscr{T}_1 \circ \mathscr{G}_2 = \mathscr{G}_1^{\circ 2} - \mathscr{G}_2^{\circ 2} \tag{2.66}$$

We thus put

$$\tilde{A}_2 = \tilde{H}_1 (C')^+ H_2^{-1}, \qquad H_2 = C' \tilde{H}_1 (C')^+ \tag{2.67}$$

to decompose $\tilde{H}_1^{-1}$ and obtain the next order integrand

$$\exp[i\langle J_2, \tilde{\mathscr{A}}_2 \psi_2 \rangle - \langle \psi_2, H_2^{-1}\psi_2 \rangle + i\langle J_2, \tilde{A}_1 Q \tilde{\psi}_1 \rangle - \langle \tilde{\psi}_1, Q^+ \tilde{H}_1^{-1} Q \tilde{\psi}_1 \rangle] \tag{2.68}$$

where we have set $\tilde{\mathscr{A}}_2 \equiv \tilde{A}_1 \tilde{A}_2$ and

$$J_2 = -\frac{1}{\sqrt{N}}[\varphi_2^2(x) - (\beta - \mathscr{T}_0 - \mathscr{T}_1)] = -\frac{1}{\sqrt{N}} :\varphi_2^2(x):_{G_2} \tag{2.69}$$

Thus we have $\beta_2 = \beta - \mathscr{T}_0(x, x) - \mathscr{T}_1(x, x)$ and if $\phi_2 \in \mathscr{K}_2$, we can carry out the integral over $\tilde{\psi}_1$ and obtain the new factor

$$\det{}^{-1/2}(Q^+ \tilde{H}_1^{-1} Q) \exp[-\mathscr{F}_2]$$

where

$$\mathscr{F}_2 \equiv \sum_{x,y} f_2(x,y)(Q^+\tilde{\mathscr{A}}_1^+ :\varphi_2^2:_{G_2})(x)(Q^+\tilde{\mathscr{A}}_1^+ :\varphi_2^2:_{G_2})(y) \tag{2.70}$$

$$f_2(x,y) \equiv \frac{1}{4N}[Q^+\tilde{H}_1^{-1}Q]^{-1}_{xy} \tag{2.71}$$

$\mathscr{F}_2$ is again small and marginal, and we neglect $\mathscr{F}_2$ as the first approximation. The analysis of $\tilde{H}_1$, H_1^{-1}, $\tilde{A}_2$ and H_2^{-1} is similar and straightforward, and is left to the reader. (See next section.)

3. THE MAIN PART OF THE RECURSION FORMULA

3.1. The Main Recursion Formula

We now discuss the approximate renormalization group flow. The point is that the large coefficient $\mathscr{G}_n \sim \beta_n$ in M_n becomes the coefficient of the zero-average fluctuation fields $Q\tilde{\psi}_n$ and the coefficient of the block spin ψ_{n+1} of next order is $O(1)$. This means the following:

Fact 1. The inner product $\varphi_{n+1}(x)\, z_n(x)$ in $\langle :\varphi_n^2:, \tilde{\mathscr{A}}_n\psi_n\rangle$ must be small, namely the fluctuations ξ_n parallel to ϕ_{n+1} are small,

Fact 2. As a result of the above, the integral by $Q\tilde{\psi}_n$ yields small terms which may be neglected, and the coefficient carried by ψ_{n+1} is $O(1)$ and then the integral by ψ_{n+1} still has an effect of order $O(1)$.

Thus the main part of the recursion formula is obtained by the following simplifications:

[A1] discard small marginal term $\langle Q^+\tilde{\mathscr{A}}_{n-1}^+ J_n,\ [Q^+\tilde{H}_{n-1}Q]^{-1}\, Q^+ \tilde{\mathscr{A}}_{n-1}^+ J_n\rangle$ which arises by the integration by $d\tilde{\psi}_n$,

[A2] discard the subtracted determinant $\det_3^{-N/2}(1+iK_n)$, $K_n = 2\mathscr{T}_n(\tilde{\mathscr{A}}_n\psi_n)/\sqrt{N}$ since $N(\mathrm{Tr}\, K_n^k)$ $(k \geqslant 3)$ are small and irrelevant (irrelevant means that these terms disappear if we iterate the block spin transformations),

[A3] discard terms (e.g., $\delta\tilde{H}_n$, etc.) which are marginal and small.

(Some notation are defined below.) They are all of order $O(\beta^{-1})$ or $O(N^{-1})$. We also neglect large field contributions. This problem will be discussed in the forthcoming paper.[13]

Then neglecting all marginal terms, on $\mathscr{K}_n$, we have

$$W_n(\phi_n, \psi_n) = \tfrac{1}{2}\langle \phi_n, G_n^{-1}\phi_n\rangle + \langle \psi_n, H_n^{-1}\psi_n\rangle - i\langle J_n, \tilde{\mathscr{A}}_n\psi_n\rangle \tag{3.1}$$

and we easily see that the kernels satisfy the following recursion relations if we neglect small non-local terms:

$$J_n(\phi_n) = J_{n-1}(A_n\phi_n) - \sqrt{N}\,\mathscr{T}_{n-1} = \sqrt{N}\left(\beta - \sum_{i=0}^{n-1} \mathscr{T}_i - \frac{1}{N}\varphi_n^2\right) \tag{3.2a}$$

$$\tilde{H}_{n-1}^{-1} = H_{n-1}^{-1} + \tilde{\mathscr{A}}_{n-1}^{+} M_{n-1} \tilde{\mathscr{A}}_{n-1}$$

$$M_{n-1} = \mathscr{T}_{n-1}^{\circ 2} + 2\mathscr{T}_{n-1} \circ \mathscr{G}_n = \mathscr{G}_{n-1}^{\circ 2} - \mathscr{G}_n^{\circ 2} \tag{3.2b}$$

$$H_n = C'\tilde{H}_{n-1}(C')^{+} = C'\{H_{n-1}^{-1} + \tilde{\mathscr{A}}_{n-1}^{+} M_{n-1} \tilde{\mathscr{A}}_{n-1}\}^{-1}(C')^{+}$$

$$\tilde{A}_n = \tilde{H}_{n-1}(C')^{+} H_n^{-1} \tag{3.2c}$$

$$\tilde{\mathscr{A}}_n = \tilde{A}_1 \cdots \tilde{A}_n \tag{3.2d}$$

with $H_0^{-1} = 0$ and $G_0 = (-\Delta + m^2)^{-1}$, where we have used

$$G_n = A_{n+1} G_{n+1} A_{n+1}^{+} + Q\Gamma_n Q^{+}, \qquad \mathscr{G}_n = \mathscr{G}_{n+1} + \mathscr{T}_n \tag{3.3}$$

These relations may look quite complicated, but one of the roles of the recursion formulas for $\tilde{A}_n$ and $\tilde{H}_n$ is an approximate derivation of $\langle \psi(x)\,\psi(y)\rangle$ through the multiscale decompositions of $G^{\circ 2}$:

$$G^{\circ 2} = \sum_{n=0}^{\infty} (\mathscr{G}_n^{\circ 2} - \mathscr{G}_{n+1}^{\circ 2}) = \sum_{n=0}^{\infty} M_n \tag{3.4}$$

Before solving these recursion formulas, we note that $G_n(x, y)$ depends only on $x - y$. Then $\mathscr{A}_n(x, y)$ depends only on $x - L^n y$, $x \in \Lambda$, $y \in \Lambda_n$ and is invariant by the simultaneous shifts $x \to x + L^n e_\mu$, $y \to y + e_\mu$ where $e_\mu = \{(1, 0), (0, 1)\}$. Then $\mathscr{G}_n(x, y)$ is invariant for $x \to x + e_\mu L^n$ and $y \to y + e_\mu L^n$. This is also the case for $\mathscr{T}_{n-1} = \mathscr{G}_{n-1} - \mathscr{G}_n$ and M_{n-1}. This means that $\tilde{H}_0^{-1}(x, y)$ is invariant for $x \to e_\mu L$ and $y \to e_\mu L$ and then $H_1(x, y)$ depends only on $x - y$. Then by induction, we see that $H_n(x, y)$ depends only on $x - y$ and $\tilde{\mathscr{A}}_n(x, y)$ depends only on $x - L^n y$. We also note that $\sum_y G_n(x, y)$ and $\sum_y G_n^{\circ 2}(x, y)$ are independent of x. Then we see that $\sum_y \mathscr{A}_n(x, y)$, $\sum_y \tilde{\mathscr{A}}_n(x, y)$ and $\sum_y M_n(x, y)$ are all independent of $x \in \Lambda$.

3.2. Solving the Recursion Formula

The properties of $\mathscr{A}_n$, $\mathscr{T}_n$ and $\mathscr{G}_n$ are well known, and we study $\tilde{A}_n$ and H_n. Since

$$(C^n \mathscr{A}_n)(x, y) = ((C')^n\, \tilde{\mathscr{A}}_n)(x, y) = \delta_{x, y}, \qquad x, y \in \Lambda_n \tag{3.5}$$

and $\mathscr{A}_n$ and $\tilde{\mathscr{A}}_n$ decrease exponentially fast, we expect

$$\mathscr{A}_n(x, y) = (A_1 \cdots A_n)(x, y) \sim \delta_{[\frac{x}{L^n}], y}$$
$$\tilde{\mathscr{A}}_n(x, y) = (\tilde{A}_1 \cdots \tilde{A}_n)(x, y) \sim L^{-2n} \delta_{[\frac{x}{L^n}], y}$$

where $[x/L^n] \in \Lambda$ is the lattice point nearest from x/L^n. These properties show that these operators are nearly diagonal. But this is not quite so, and $\mathscr{A}_n(x, y)$ has a rather long tail around const. L^n. The following two theorems mean that $\tilde{\mathscr{A}}_n$ has no tail for large β_n.

Theorem 3. The transformation matrix $\tilde{A}_n(x, y)$ is the function of $x - Ly$ only ($x \in \Lambda_{n-1}$, $y \in \Lambda_n$) and almost diagonal for $\beta_n \gg L^2$:

$$\tilde{A}_n(x, y) = \frac{1}{L^2} \delta_{[x/L], y} + \frac{1}{\beta_n} \delta \tilde{A}_n(x, y)$$

$$|\delta \tilde{A}_n(x, y)| \leqslant O\left(\exp\left[-c \left| \frac{x}{L} - y \right| \right] \right)$$

where $\beta_n = \mathscr{G}_n(x, x)$, $c = O(1)$ is a constant independent of L and β,

$$\sum_{y \in \Lambda_n} \delta \tilde{A}_n(x, y) = 0, \qquad \sum_{\zeta \in \square_{Lx}} \delta \tilde{A}_n(\zeta, y) = 0 \tag{3.6}$$

and $\square_{Lx}$ is the square of size $L \times L$ center at $Lx \in \Lambda$. The Hamiltonian kernel H_n^{-1} of ψ_n is strictly positive and bounded from below and above uniformly in β:

$$|H_n^{-1}(x, y)| < \text{const.} \exp[-c\,|x - y|] \tag{3.7}$$

where c is a positive constant independent of L and β. $H_n^{-1}(x, y)$ depends only on $x - y$.

Theorem 4. The transformation matrix $\tilde{\mathscr{A}}_n(x, y)$ is the function of $x - L^n y$ only, almost diagonal and has tail but the tail becomes small as β_n becomes large compared with L^2:

$$\tilde{\mathscr{A}}_n(x, y) = \frac{1}{L^{2n}} \delta_{[\frac{x}{L^n}], y} + \frac{1}{\beta_n L^{2(n-1)}} \delta \tilde{\mathscr{A}}_n(x, y) \tag{3.8a}$$

$$|\delta \tilde{\mathscr{A}}_n(x, y)| < \text{const.} \exp\left[-c \left| \frac{x}{L^n} - y \right| \right] \tag{3.8b}$$

where c is a positive constant independent of L and β. Furthermore

$$\sum_{y \in \Lambda_n} \delta\tilde{\mathscr{A}}_n(x, y) = 0, \qquad \sum_{\zeta \in \square^n_{L^n x}} \delta\tilde{\mathscr{A}}_n(\zeta, y) = 0 \tag{3.9}$$

where $\square^n_{L^n x}$ is the square of size $L^n \times L^n$ center at $L^n x \in \Lambda$, $x \in \Lambda_n$.

Remark 4. We can say that these theorems yield an alternative of the multiscale decomposition of $(G^{\circ 2})^{-1}$. Though $G(x, y) \sim \beta - (2\pi)^{-1} \log(|x-y|)$ is of long-range, G^{-1} is of short range: $G^{-1}(x, y) = (-\Delta + m^2)(x, y) = 0$ unless $|x-y| \leqslant 1$. Since $\psi = \sum_{n=0}^{\infty} \tilde{\mathscr{A}}_n Q \tilde{\psi}_n$, Theorem 4 means that

$$\langle \psi(x)\, \psi(y) \rangle \sim \sum_{n=0}^{\infty} \left(\tilde{\mathscr{A}}_n Q \frac{1}{2Q^+ \tilde{H}_n^{-1} Q} Q^+ \tilde{\mathscr{A}}_n^+ \right)(x, y) \sim \frac{1}{\beta(x, y)\, |x-y|^4} \tag{3.10}$$

for $|x-y| < m^{-1}$, where $\beta(x, y) = \beta_n$ for x, y such that $L^n \leqslant |x-y| < L^{n+1}$. (3.10) is close to $[G^{\circ 2}]^{-1}(x, y)$ but is slightly different. The explicit calculation of $[G^{\circ 2}]^{-1}$ is given in Appendix A. We believe that our multiscale calculation (3.10) is a better approximation for $\langle \psi(x)\, \psi(y) \rangle$, but the investigation remains.

Theorem 4 may be regarded as a generalization of the following equality obtained by the standard contour integral, where $\zeta_i \in R^2$ and $\zeta_0 = y$, $\zeta_{n+1} = x$ (see Appendix D for proof):

$$\int \exp\left(-\alpha \sum_{i=0}^{n} \left| \frac{\zeta_{i+1}}{L} - \zeta_i \right| \right) \prod_{i=1}^{n} d^2\zeta_i$$

$$= \left(\frac{2\pi}{\alpha^2} \right)^n \left(1 + O\left(\frac{1}{L} \right) \right) \exp\left(-\alpha \left| \frac{x}{L^{n+1}} - y \right| \right) \tag{3.11}$$

We prove these theorems simultaneously by induction.

(*Step 1*). Some properties (dependence on x, y, etc.) of $\tilde{\mathscr{A}}_n(x, y)$ and $H_n(x, y)$ are already established. We note that $\sum_y \mathscr{G}_n(x, y)$, $\sum_{y \in \Lambda} \mathscr{G}_n^{\circ 2}(x, y)$ and

$$\sum_{y \in \Lambda} M_n(x, y) \equiv \sum_{y \in \Lambda} (\mathscr{G}_n^{\circ 2} - \mathscr{G}_{n+1}^{\circ 2})(x, y)$$

are constants independent of x ($n=0, 1,\ldots$). Let P_0 be the projection onto the set of block-wise constant functions, and set $P_1 = 1 - P_0$, see (2.38). Then

$$\sum_{y \in \Lambda_1} (P_1 M_0 P_0)(x, Ly) = \sum_{y \in \Lambda_1} [P_1(\mathscr{G}_0^{\circ 2} - \mathscr{G}_1^{\circ 2})\, P_0](x, Ly) = 0 \tag{3.12}$$

For later use, we set $\delta_\square = L^2 C^+ = (C')^+$, namely we define $\delta_\square$ by $\delta_\square(x, y) = \delta_{[x/L],\, y}$, $x \in \Lambda_{n-1}$, $y \in \Lambda_n$. Let M be a map $R^{\Lambda_n} \to QR^{\Lambda_{n-1} \setminus L\Lambda_n}$ such that $|M(x, y)| \leqslant \exp[-c\,|x/L - y|]$, $c = O(1) > 0$ and $M(x, y)$ depends only on $x - Ly$ ($\sum_y M(x, y)$ is independent of x). Then it follows that $\sum_y M(x, y) = 0$. Let us consider the following sum

$$\sum_{\zeta \in \square^k_{L^k y}} M(x, \zeta) = \sum_{\zeta \in \square^k} M(x, L^k y + \zeta) = (M\delta^k_\square)(x, y) \tag{3.13}$$

where $\square^k$ is the square of size $L^k \times L^k$ center at the origin. Since $\sum_\zeta M(x, \zeta) = 0$, we have

$$(M\delta^k_\square)(x, y) = -\sum_{\zeta \in (\square^k_{L^k y})^c} M(x, \zeta) \tag{3.14}$$

Thus for both cases of $x \in \square^k_{L^k y}$ and $x \notin \square^k_{L^k y}$, we have

$$\left|\sum_{\zeta \in \square^k} M(x, L^k y + \zeta)\right| = |(M\delta^k_\square)(x, y)| \leqslant c_1 \exp\left[-c \operatorname{dist}\left(\frac{x}{L}, \partial\square^k_{L^k y}\right)\right] \tag{3.15}$$

where $\partial\square^k_{L^k y} \subset \Lambda_1$ is the boundary of the square $\square^k_{L^k y}$. Then

$$\begin{aligned} &\operatorname{dist}\left(\frac{x}{L}, \partial\square^k_{L^k y}\right) \geqslant \mathrm{Max}_{i=1,2}\left\{\left|\frac{x_i}{L} - L^k\left(y_i \pm \frac{1}{2}\right)\right|\right\} \\ &|(\delta_\square^{n-k} M \delta^k_\square)(x, y)| \leqslant O(1) \exp\left[-c \operatorname{dist}\left(\left[\frac{L^k x}{L^n}\right], \partial\square^k_{L^k y}\right)\right] \end{aligned} \tag{3.16}$$

If $|xL^{-n} - y| > O(1)$, $x \in \Lambda$, $y \in \Lambda_n$, then $|(\delta_\square^{n-k} M\delta^k_\square)(x, y)|$ decreases rapidly. More generally, we prove in Appendix B (Lemma B.2) that

$$|(M\delta^k_\square)(x, y)| \leqslant O(L^{-k}) \exp[-c'\,|x/L - L^k y|] \tag{3.17}$$

where $0 < c' \sim c$. Intuitively speaking, $M(x, y)$ is averaged by y and tends to a constant ($=0$ this case) independent of x as $k \to \infty$.

(*Step 2*). We substitute $\tilde{\mathscr{A}}_n$ into $\tilde{H}_n^{-1}$ and obtain

$$\begin{aligned}\tilde{H}_n^{-1} &= H_n^{-1} + 2\beta_{n+1} Q\Gamma_n Q^+ + C^n[\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})](C^+)^n \\ &\quad + \frac{2\beta_{n+1}}{\beta_n L^{2(n-1)}} [C^n \mathscr{T}_n \delta\tilde{\mathscr{A}}_n + \delta\tilde{\mathscr{A}}_n^+ \mathscr{T}_n (C^+)^n] \\ &\quad + \frac{1}{\beta_n L^{2(n-1)}} [C^n(\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1}))\, \delta\tilde{\mathscr{A}}_n + (\text{h.c.})] \\ &\quad + \frac{1}{\beta_n^2 L^{4(n-1)}}\, \delta\tilde{\mathscr{A}}_n^+ M \delta\tilde{\mathscr{A}}_n \end{aligned} \tag{3.18}$$

where we put $\mathscr{G}_{n+1} = \beta_{n+1} + (\mathscr{G}_{n+1} - \beta_{n+1})$ and used

$$(\tilde{\mathscr{A}}_n^+ \mathscr{A}_n)(x, y) = \delta_{x,y} + \sum_{\zeta} \frac{1}{\beta_n L^{2(n-1)}}\, \delta\tilde{\mathscr{A}}_n(\zeta, x)\, \mathscr{A}_n(\zeta, y) \tag{3.19}$$

The operator $[C^n(\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1}))(C^+)^n](x, y)$ is bounded uniformly in β and decreases like $O(\exp[-|x-y|])$ since $|\mathscr{G}_{n+1}(x, y) - \beta_{n+1}| < O(1)$ for $|x-y| < m^{-1}$. The term in the second line in (3.18) contains $\delta\tilde{A}_n$ and is order of $O(1)$. This is written

$$\begin{aligned}&\frac{2\beta_{n+1}}{\beta_n L^{2(n-1)}} [C^n \mathscr{T}_n \delta\tilde{\mathscr{A}}_n + \delta\tilde{\mathscr{A}}_n^+ \mathscr{T}_n (C^+)^n] \\ &\qquad = \frac{2\beta_{n+1}}{\beta_n L^{2(n-1)}} [Q\Gamma_n Q^+ \mathscr{A}_n^+ \delta\tilde{\mathscr{A}}_n + \delta\tilde{\mathscr{A}}_n^+ \mathscr{A}_n Q\Gamma_n Q^+]\end{aligned} \tag{3.20}$$

Let P_0 be the projection operator onto the set of block-wise constant functions, and put $P_1 = 1 - P_0$, see Eq. (2.38). Let $E = \tilde{H}_n^{-1}$ and set $E_{ij} = P_i E P_j$. Then we have

$$\begin{aligned} E_{00} &= P_0[H_n^{-1} + C^n[\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})](C^+)^n + O(\beta_n^{-1})]\, P_0 \\ E_{11} &= P_1[2\beta_{n+1} Q\Gamma_n Q^+ + O(1) + O(\beta_n^{-1})]\, P_1, \\ E_{10} &= P_1[H_n^{-1} + C^n[\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})](C^+)^n]\, P_0 \\ &\quad + P_1\left[\frac{2\beta_{n+1}}{\beta_n L^{2(n-1)}} Q\Gamma_n Q^+ \mathscr{A}_n^+ \delta\tilde{\mathscr{A}}_n + O(\beta_n^{-1})\right] P_0 \end{aligned}$$

and $\tilde{H}_n = \{\tilde{H}_{ij}\}$ where

$$\tilde{H}_{00} = (E_{00} - E_{01}E_{11}^{-1}E_{10})^{-1}, \qquad \tilde{H}_{10} = -E_{11}^{-1}E_{10}\tilde{H}_{00}$$

$$\tilde{H}_{11} = (E_{11} - E_{10}E_{00}^{-1}E_{01})^{-1}, \qquad \tilde{H}_{01} = -E_{00}^{-1}E_{01}\tilde{H}_{11}$$

Therefore we have

$$\tilde{A}_{n+1}(x, y) = \frac{1}{L^2}\,\delta_{[x/L],\,y} - \left(\frac{1}{E_{11}}\,E_{10}C^+\right)(x, y) \tag{3.21}$$

$$H_{n+1} = C'\,\frac{1}{E_{00} - E_{01}E_{11}^{-1}E_{10}}\,C'^+ \tag{3.22}$$

(*Step 3*). From (3.21), we have the recursion formula

$$\delta\tilde{A}_{n+1} = -\beta_{n+1}\,\frac{1}{E_{11}}\,E_{10}C^+ \equiv D_{n+1} + (b_{n+1} + \varepsilon_n^{(1)})\,\delta\tilde{\mathscr{A}}_n C^+ \tag{3.23}$$

where

$$D_{n+1} = B_{n+1} + \varepsilon_n^{(0)} \tag{3.24a}$$

$$B_{n+1} \equiv -\frac{\beta_{n+1}}{E_{11}}\,P_1[H_n^{-1} + C^n[\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})](C^+)^n]\,C^+ \tag{3.24b}$$

$$b_{n+1} \equiv -\frac{2\beta_{n+1}^2}{\beta_n L^{2(n-1)}E_{11}}\,P_1 Q\Gamma_n Q^+\mathscr{A}_n^+ \tag{3.24c}$$

$$\varepsilon_n^{(0)} \equiv -\frac{\beta_{n+1}}{\beta_n L^{2(n-1)}E_{11}}\,P_1[C^n[\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})](C^+)^n]\,C^+ \tag{3.24d}$$

$$\varepsilon_n^{(1)} \equiv -\frac{\beta_{n+1}}{\beta_n^2 L^{4(n-1)}E_{11}}\,P_1\delta\tilde{\mathscr{A}}_n^+ M_n \tag{3.24e}$$

and then

$$\begin{aligned} B_{n+1} &= -\left(1 + O\left(\frac{1}{\beta_{n+1}}\right)\right)\frac{1}{2P_1 Q\Gamma_n Q^+ P_1} \\ &\quad \times P_1(H_n^{-1} + C^n[\mathscr{T}_n^{\circ 2} + 2\mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})](C^+)^n)\,C^+ \\ b_{n+1} &= -\frac{\beta_{n+1}}{\beta_n L^{2(n-1)}}\left(1 + O\left(\frac{1}{\beta_{n+1}}\right)\right)\frac{1}{P_1 Q\Gamma_n Q^+ P_1}\,P_1 Q\Gamma_n Q^+\mathscr{A}_n^+ \\ &= -\frac{\beta_{n+1}}{\beta_n L^{2(n-1)}}\left(1 + O\left(\frac{1}{\beta_{n+1}}\right)\right)P_1\mathscr{A}_n^+. \end{aligned}$$

since $P_0 C^+ = C^+$ and $Q^+ = Q^+ P_1$. Note that $[C^n \mathscr{T}_n^{\circ 2} (C^+)^n](x, y)$ and $[C^n \mathscr{T}_n \circ (\mathscr{G}_{n+1} - \beta_{n+1})(C^+)^n](x, y)$ in D_{n+1} are bounded by $\exp[-O(1/L)\,|x-y|]$.

Similarly from the definition of $\tilde{\mathscr{A}}_{n+1}$, we have

$$\delta\tilde{\mathscr{A}}_{n+1} = \frac{\beta_{n+1}}{\beta_n}\,\delta\tilde{\mathscr{A}}_n \delta_\square + \left(\delta_\square^n + \frac{L^2}{\beta_n}\,\delta\tilde{\mathscr{A}}_n\right)\delta\tilde{A}_{n+1} \tag{3.25}$$

Then from (3.23) and (3.25), we obtain the recursion relations for $\{\delta\tilde{\mathscr{A}}_n\}$:

$$\delta\tilde{\mathscr{A}}_{n+1} = \delta_\square^n D_{n+1} + \tau_n \delta\tilde{\mathscr{A}}_n \delta_\square + f_n(\delta\tilde{\mathscr{A}}_n) \tag{3.26}$$

where

$$\tau_n = \frac{\beta_{n+1}}{\beta_n} + \frac{1}{L^2}\,\delta_\square^n b_{n+1} = \frac{\beta_{n+1}}{\beta_n}\left(1 - \frac{1}{L^{2n}}\,\delta_\square^n P_1 \mathscr{A}_n^+\right) + \tau'_n \tag{3.27}$$

$$\tau'_n = O(\beta_n^{-1}) \sim \frac{1}{\beta_n L^n}\,\delta_\square^n\,\frac{1}{Q\Gamma_n Q^+}\,O(1)\,P_1 \mathscr{A}_n^+ \tag{3.28}$$

$$f_n(\delta\tilde{\mathscr{A}}_n) = \frac{L^2}{\beta_n}\,\delta\tilde{\mathscr{A}}_n D_{n+1} + \left[\left(\delta_\square^n + \frac{L^2}{\beta_n}\,\delta\tilde{\mathscr{A}}_n\right)\varepsilon_n^{(1)} + \frac{1}{\beta_n}\,\delta\tilde{\mathscr{A}}_n b_{n+1}\right]\delta\tilde{\mathscr{A}}_n \delta_\square \tag{3.29}$$

Then $f_n(\delta\tilde{\mathscr{A}}_n) = O(\beta_n^{-1})$ satisfies $\sum_y f_n(\delta\tilde{\mathscr{A}}_n)(x, y) = 0$. Thus, from (3.26) we have

$$\delta\tilde{\mathscr{A}}_{n+1} = \Xi_n + f_n(\delta\tilde{\mathscr{A}}_n) + \tau_n f_{n-1}(\delta\tilde{\mathscr{A}}_{n-1})\,\delta_\square + \cdots + \tau_n \cdots \tau_2 f_1(\delta\tilde{\mathscr{A}}_1)\,\delta_\square^n \tag{3.30}$$

$$\Xi_n = \tau_n \cdots \tau_1 \delta\tilde{A}_1 \delta_\square^n + \delta_\square^n D_{n+1} + \tau_n \delta_\square^{n-1} D_n \delta_\square \cdots + \tau_n \cdots \tau_2 \delta_\square D_2 \delta_\square^{n-1} \tag{3.31}$$

(*Step 4*). Thus if $|x/L^n - y| > O(1)$, then (3.16) can be applied to the right hand sides, and we complete the proof. We first note $\beta_{n+1}/\beta_n < 1$, and set

$$\kappa_n^{(0)} = \frac{1}{L^{2n}}\,\delta_\square^n P_0 \mathscr{A}_n^+, \qquad \kappa_n^{(1)} = 1 - \frac{1}{L^{2n}}\,\delta_\square^n \mathscr{A}_n^+ \tag{3.32}$$

so that $\tau_n = \frac{\beta_{n+1}}{\beta_n}(\kappa_n^{(0)} + \kappa_n^{(1)}) + \tau'_n$. Then using the definition $\mathscr{A}_n^+ = G_n^{-1} C^n G_0$ and $P_0 = L^2 C^+ C$, we can easily confirm the following relations:

$$\kappa_n^{(i)} \kappa_n^{(j)} = \delta_{ij} \kappa_n^{(i)}, \qquad \kappa_n^{(i)} \kappa_{n-1}^{(j)} = \delta_{ij} \kappa_n^{(i)} \kappa_{n-1}^{(i)}, \qquad i, j = 0, 1 \tag{3.33}$$

$$\kappa_m^{(0)} \kappa_n^{(1)} = 0, \qquad \kappa_m^{(1)} \kappa_n^{(1)} = \kappa_n^{(1)}, \qquad \text{for} \quad m > n \tag{3.34}$$

Thus we have

$$\tau_n \tau_{n-1} \cdots \tau_k = \frac{\beta_{n+1}}{\beta_k} [\kappa_n^{(0)} \cdots \kappa_k^{(0)} + \kappa_k^{(1)}] + (\text{terms containing } \tau') \tag{3.35}$$

where

$$\kappa_n^{(0)} \cdots \kappa_k^{(0)} = \delta_\square^{n+1} C^{n-k+1} \frac{1}{L^{2k}} \mathscr{A}_k^+ \tag{3.36}$$

$$\kappa_k^{(1)} \delta_\square^{k-1} = \delta_\square^{k-1} (1 - C^+ G_n^{-1} C G_{n-1}) \tag{3.37}$$

and then the product $\kappa_n^{(0)} \cdots \kappa_k^{(0)}$ yields $L^{-2(n-k+1)}$. Another factor $\kappa_k^{(1)} \delta_\square^{k-1}$ acts on

$$(D_k \delta_\square^{n+1-k})(x, y) = \sum_{\zeta \in \square^{n+1-k}} D_k(x, L^{n+1-k} y + \zeta), \qquad x \in \Lambda_k, \quad y \in \Lambda_n$$

which is slowly varying since the right hand side contains

$$\sum_{\zeta \in \square^{n+1-k}} \exp[ip\zeta] = \prod_{i=1}^{2} \frac{\sin(L^{n+1-k} p_i/2)}{\sin(p_i/2)}$$

$\kappa_k^{(1)} \delta_\square^{k-1} C^+ = 0$ means that $\kappa_k^{(1)} \delta_\square^{k-1} = 0$ on the set of (block-wise) constant functions (block size is $L \times L$). Then by (3.17), we have

$$(1 - C^+ G_n^{-1} C G_{n-1})(D_k \delta_\square^{n+1-k}) = O(L^{-(n+1-k)}) \tag{3.38}$$

The terms containing τ_k' are convergent and small.

Thus Ξ_n is bounded uniformly in n if $\beta_n \sim \beta - (\log L^n)/2\pi \gg L^2$. The main term in Ξ_n is $\delta_\square^n D_{n+1}$. We choose $\beta_n > > L^2$ so that $f_n(\delta \tilde{\mathscr{A}}_n)$ are of order $O(\beta_n^{-1})$. Since $f_n(\delta \tilde{\mathscr{A}}_n)$ satisfies (3.17), we choose L so large that $\delta \tilde{\mathscr{A}}_n \delta_\square$ is sufficiently small getting the factor of order $O(L^{-1})$:

$$\delta \tilde{\mathscr{A}}_{n+1} = \Xi_n + \min \left\{ O(L^{-1}), \frac{L^2}{\beta_n} \right\} O(\delta \tilde{\mathscr{A}}_n) \tag{3.39}$$

The remaining properties of $\delta \tilde{\mathscr{A}}_{n+1}$ and H_{n+1}^{-1} are clear from the expression of $\delta \tilde{\mathscr{A}}_{n+1}$. Q.E.D.

4. APPROXIMATE FLOW OF THE RG

The renormalization group flow can be obtained from these recursion formulas with the help of Theorem 4. The formulas for J_n and G_n are closed and then solved directly by (3.3) since $\beta = G_0(x, x) = \mathscr{G}_0(x, x)$. Thus J_n is explicitly obtained:

$$J_n(x) = \sqrt{N}\left[\mathscr{G}_n(x, x) - \frac{\varphi_n^2(x)}{N}\right] = -\frac{1}{\sqrt{N}} :\varphi_n^2(x):_{G_n} \tag{4.1}$$

Since $G_0(x) \sim \beta - (2\pi)^{-1} \log(1+|x|)$ for $|x| < m^{-1}$ and $G_0(x) \sim c_1 \exp[-c_2 m\,|x|]$ for $|x| > m^{-1}$ (c_1 and $c_2 \sim 1$ are positive constants), we see

1. $G_n(x, y) \sim \beta - (2\pi)^{-1} \log L^n(1+|x-y|)$, if $L^n\,|x-y| < m^{-1}$
2. $G_n(x, y) \sim L^{-2n} m^{-2} \delta_{xy}$, if $L^n m > 1$

and then we have

$$\mathscr{G}_n(x, x) \sim \beta_n \equiv \begin{cases} \beta - n(2\pi)^{-1} \log L, & \text{if} \quad L^n m \ll 1 \\ L^{-2n} m^{-2}, & \text{if} \quad L^n m > 1 \end{cases} \tag{4.2}$$

We can regard this quantity as the average of the field $\varphi_n^2/N \sim \phi_n^2/N$, as is seen by the integration over ψ_n. Thus we can regard this as the approximate flow of the renormalization group of the model.

The flow (4.2) of β_n is realized not only in the hierarchical model of Dyson–Wilson type but also in the hierarchical model of Gallavotti type. (This type of flow of β_n is obtained in most of reasonable approximative calculations of the two-dimensional $O(N)$ spin model.)

How about the effective interactions? To see it, we need Theorems 3 and 4 which mean that $\tilde{\mathscr{A}}_n \sim L^{-n} \delta_{[x/L^n], y}$ and $H_n(x, y) \sim \delta_{x, y}$:

$$H_{n+1}^{-1}(x, y) \equiv [C' \tilde{H}_n C'^{+}]^{-1}(x, y) \sim \text{const.}\ \delta_{x, y} \tag{4.3}$$

$$\frac{i}{\sqrt{N}} \langle :\varphi_n^2:, \tilde{\mathscr{A}}_n \psi_n \rangle \sim \frac{i}{\sqrt{N}} \langle :\phi_n^2:, \psi_n \rangle \tag{4.4}$$

This means that the integration over ψ_n yields the double well potential

$$\mathscr{V}_n \sim \frac{1}{N} (\phi_n^2(x) - N\beta_n)^2 \sim \beta(|\phi_n(x)| - (N\beta_n)^{1/2})^2 \tag{4.5}$$

But we have to mention that this may have to be taken with a grain of salt since this integration is justified for small $|:\varphi_n^2:|$ since we must assume that $|\psi| < N^{1/2}$ to expand the determinant. Even so, this means that the parallel component of the fluctuation field ξ_n must be small, and thus we expect that the constraint of type (2.52a) is reproduced. This flow is close to that of the hierarchical model of Dyson–Wilson type[4, 8, 12, 19] with large N.

5. CONCLUSION

In this paper, we have shown that the system is close to that described by the hierarchical model advocated by Wilson.[19] The reason for this is that the auxiliary fields ψ_n can be sharply decomposed into the block spins ψ_{n+1} and the fluctuations $\tilde{\psi}_n$ when $\beta_n > L^2$. This reflects the fact that the spin fluctuation fields $\xi_n(x)$ are almost orthogonal to the block spins $\phi_{n+1}(x)$. An immediate consequence from this is that $|{:}\phi_n^2(x){:}_{G_n}|$ must be small on $\mathscr{K}_n$, which will be used in our forthcoming paper.[12] Another consequence is that the main contribution of the ψ integral comes from $|\psi| < \text{const.}\beta^{-1/2}$ since the main contribution comes from $|\tilde{\psi}_n(x)| < \beta_n^{-1/2}$ and

$$|\psi(x)| = \left|\sum_n (\tilde{\mathscr{A}}_n Q \tilde{\psi}_n)(x)\right| \sim \sum \frac{1}{L^{2n}} \beta_n^{-1/2}$$

Thus, we expect that corrections to the correlation length $\xi \sim m^{-1} \sim e^{2\pi\beta}$ are small.

The effective interaction term $\mathscr{V}_n$ is calculated in some hierarchical models. The hierarchical model approximation advocated by Gallavotti[6, 9] yields

$$\frac{1}{4N\beta_n} (\phi_n^2(x) - N\beta_n)^2 \sim (|\phi_n(x)| - (N\beta_n)^{1/2})^2 \tag{5.1}$$

which should be compared with (4.5).

Remark 5. The hierarchical model of Dyson–Wilson type[4, 19] is obtained by replacing spin variables $\{\phi(x);\, x \in I\}$ in a block I by $\kappa\phi_1(x_I) \pm z$ where $\phi_1(x_I)$ is the block spin (average of spins) on I, $\pm z$ are zero-average fluctuations in I and κ is a suitable factor. On the other hand, the hierarchical model of Gallavotti type[9, 6] is obtained by replacing spin variables $\{\phi(x);\, x \in I\}$ by $\kappa\phi_1(x_I) + z$. In both approximations, the variable z is a Gaussian random variable localized in I, but in the latter approximation, the sum of the fluctuations z is not zero. Since we have used the block-spin transformation of Kadanoff–Wilson type in which the sum of the fluctuations is zero, it may be natural that we have obtained the flow of Dyson–Wilson type.

APPENDIX A. INVERSE OF $G^{\circ 2}$

Set $\tilde{G}(p) = [m^2 + 2\sum (1 - \cos p_i)]^{-1}$. Then the Fourier transform $T(p)$ of $G^{\circ 2}$ is given by

$$T(p) = \int \tilde{G}(k)\, \tilde{G}(p-k)\, \frac{d^2k}{(2\pi)^2}$$

where $k \in [-\pi, \pi]^2 \equiv I$. We are interested in p such that $m^{-1} \ll |p| = O(1) \leqslant \pi$. Set $I_0 = \{k \in I;\, |k| < |p|/2\}$, $I_1 = \{k \in I;\, |p-k| < |p|/2\}$ and $I' = I \setminus (I_0 \cup I_1)$, and let T_0, T_1 and T' be the contributions to T from the regions I_0, I_1 and I', respectively. Obviously T', the contribution from I' is less than $[m^2 + 2\sum(1-\cos(p_i/2))]^{-1} = O(1)$ since both $|k|$ and $|p-k|$ are larger than $|p/2|$. On the other hand, using $\log m^2 = -4\pi\beta + O(1)$, we have

$$T_0(p) = \int_{I_0} \frac{1}{[m^2 + 2\sum(1-\cos k_i)][m^2 + 2\sum(1-\cos(p_i - k_i))]} \frac{d^2k}{(2\pi)^2}$$
$$= \frac{\beta}{[m^2 + 2\sum(1-\cos p_i)]} \tilde{T}_0(p) \tag{A1}$$

where $\tilde{T}_0(p) \sim 1$ is a slowly varying even function of p defined in the obvious way. Thus

$$T(p)^{-1} = \frac{m^2 + 2\sum(1-\cos p_i)}{2\beta} \left\{ \frac{1}{\tilde{T}_0(p)} + O(\beta^{-1}) \right\} \tag{A2}$$

On the lattice space, $\sum 2(1-\cos p_i)$ is the lattice Laplacian, and $[\tilde{T}_0]^{-1}$ becomes a function which decreases like $O(|x-y|^{-2})$ on Z^2. Then the conclusion follows. (This is an outline of the calculation, and the remaining part is left to the treader.)

APPENDIX B. RENORMALIZATION GROUP ANALYSIS

We here briefly sketch the block-spin procedure following Gawedzki and Kupiainen in the form applied to the present system.[7] The boson field $\{\phi_n(x);\, x \in \Lambda_n\}$ is written in terms of spin variables $\{\phi_{n+1}\}$ of next distance scale and fluctuation fields $\{\xi_n(x);\, x \in \Lambda_n \setminus L\Lambda_{n+1}\}$:

$$\phi_n(x) = (A_{n+1}\phi_{n+1})(x) + (Q\xi_n)(x) \tag{B1}$$

where $A_{n+1}\colon R^{\Lambda_{n+1}} \to R^{\Lambda_n}$ is given by

$$A_{n+1}(x, y) = (G_n C^+ G_{n+1}^{-1})(x, y) \tag{B2}$$

Substituting these, we find

$$\langle \phi_n, G_n^{-1}\phi_n \rangle = \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} \rangle + \langle \xi_n, Q^+ G_n^{-1} Q\xi_n \rangle \tag{B3}$$

Thus $\{\xi_n(x); x \in \Lambda_n \setminus L\Lambda_{n+1}\}$ are Gaussian random variables of zero mean and covariance Γ_n, where

$$\Gamma_n(x, y) = (Q^+ G_n^{-1} Q)^{-1} = R(G_n - G_n C^+ G_{n+1}^{-1} C G_n)\, R^+ \tag{B4}$$

with R being the projection $R^\Lambda \to R^{\Lambda \setminus L\Lambda_1}$. $\Gamma_n(x, y)$ decays exponentially fast no matter how small m is. In fact for $n = 0$

$$\begin{aligned}(G_0 - G_0 C^+ G_1^{-1} C G_0)(x, y)\\ = \int \frac{d^2 p}{(2\pi)^2} \left[\tilde{G}_0(p) - \tilde{G}_0(p) \frac{\tilde{G}_1^{-1}(Lp)}{L^2} \tilde{G}_0(p)\, h^2(p) \right] e^{ip(x-y)}\end{aligned} \tag{B5}$$

where

$$\tilde{G}_0(p) = \left[m^2 + 2 \sum (1 - \cos p_i) \right]^{-1} \tag{B6a}$$

$$\tilde{G}_1(p) = \sum_{k_i = -(L-1)/2}^{(L-1)/2} \tilde{G}_0 \left(\frac{p + 2\pi k}{L} \right) h^2 \left(\frac{p + 2\pi k}{L} \right) \tag{B6b}$$

$$h(p) = \prod_i \frac{\sin(Lp_i/2)}{L \sin(p_i/2)} \tag{B6c}$$

In (B5), the function inside of $[\cdots]$ is analytic in $|\mathrm{Im}\, p_i| < O(1/L)$ and the singularity of $\tilde{G}_0(p)$ at $p^2 = -m^2$ is canceled. Thus Γ_0 has uniform exponential decay with decay rate $O(1/L)$. This is same for $n \geqslant 2$:

$$|\Gamma_n(x, y)| \leqslant c_0 \exp[-c\, |x - y|], \qquad c = O(1/L), \quad x, y \in \Lambda_n \tag{B7}$$

We conversely have

$$G_0(x, y) = \sum_{n=0}^{\infty} \mathcal{T}_n(x, y), \qquad \mathcal{T}_n = \mathcal{A}_n Q \Gamma_n Q^+ \mathcal{A}_n^+ \tag{B8}$$

where

$$\mathcal{A}_n(x, y) \equiv A_1 \cdots A_n(x, y) = G_0 (C^+)^n\, G_n^{-1}(x, y) \tag{B9}$$

and $x \in \Lambda$ (lattice width $= 1$), $y \in \Lambda_n$ (lattice width $= 1$) and $\mathcal{A}_0 = 1$. (This notation is different from that in ref. 8 where the first argument x in $\mathcal{A}_n(x, y)$ stands for the point in $L^{-n}\Lambda$.)

Obviously ϕ_n has the mass $m_n = L^n m$ by the dimensional reason, and we have shown that ξ_n has the mass square of order $O(L^{-1}) + m_n^2 \geqslant O(1)$.

One easily finds that $A_1(x, y) = \mathscr{A}_1(x, y)$ is given by

$$\sum_{k_i=-(L-1)/2}^{(L-1)/2} \int \frac{d^2p}{(2\pi)^2} \exp\left[ip\left(\frac{x}{L} - y\right) + \frac{2i\pi kx}{L}\right]$$
$$\times h\left(\frac{p+2\pi k}{L}\right)\frac{1}{L^2}\tilde{G}_0\left(\frac{p+2\pi k}{L}\right)\tilde{G}_1^{-1}(p) \tag{B10}$$

and $\mathscr{A}_n(x, y)$ is obtained by replacing L by L^n. The sum over (k_1, k_2) comes from the change of the variable $p_i \in (-\pi, \pi) \to (p_i + 2\pi k_i)/L$, $k_i = 0$, $\pm 1, \ldots, \pm (L-1)/2$. The function $L^2 \tilde{G}_0^{-1}[(p+2\pi k)/L]\, \tilde{G}_1^{-1}(p)$ is analytic and bounded in the strip region $|\mathrm{Im}\, p_i| < O(1/L)$ and the singularity at $p = 0$ is again absent. The sum over k is absolutely convergent and the first small k_i's are enough for the estimate.

Lemma B.1. The following bounds hold:

$$|\mathscr{A}_n(\zeta, x)| \leqslant O(1) \exp\left[-c\left|\frac{\zeta}{L^n} - x\right|\right] \tag{B11a}$$

$$|\mathscr{A}_n(\zeta, x) - \mathscr{A}_n(\xi, x)| \leqslant O(1) \frac{|\zeta - \xi|}{L^n}\left(\exp\left[-c\left|\frac{\zeta}{L^n} - x\right|\right] + \exp\left[-c\left|\frac{\xi}{L^n} - x\right|\right]\right) \tag{B11b}$$

$$|\mathscr{T}_n(x, y)| \leqslant O(1) \exp\left[-\frac{c}{L^{n+1}}|x - y|\right] \tag{B11c}$$

$$|\mathscr{T}_n(x, y) - \mathscr{T}_n(x', y)| \leqslant O(1) \frac{|x - x'|}{L^n}\left(\exp\left[-\frac{c}{L^{n+1}}|x - y|\right]\right.$$
$$\left. + \exp\left[-\frac{c}{L^{n+1}}|x' - y|\right]\right) \tag{B11d}$$

where $\zeta, \xi \in \Lambda$, $x, y \in \Lambda_n$ in the first, second equations and $x, x', y \in \Lambda$ in the third and the final equations. The constants $c's$ in these equations are chosen independent of L.

Lemma B.2. Let $M(x, y) = (1 - \Delta)^{-1}(x, y)$ and put $(M\delta_\square)(x, y_1) \equiv \sum_{\xi \in \square} M(x, Ly_1 + \xi)$. Then

$$|(M\delta_\square)(x + e_\mu, y_1) - (M\delta_\square)(x, y_1)| \leqslant \frac{c_1}{L} \exp[-c_2 |x - Ly|] \tag{B12}$$

where $c_i > 0$ are constants independent of L.

Lemma B.1 is more or less clear from our arguments and is well known. See also ref. 7. Then we prove Lemma B.2 which can be extended to the form used in the proofs of Theorems 3 and 4. Note that

$$(M\delta_{\square})(x, y_1) = \int \exp[ipL(x_1 - y_1)] \frac{e^{ip\zeta} L^2 h(p)}{1 + 2\sum (1 - \cos p_i)} \prod \frac{dp_i}{2\pi} \tag{B13}$$

$$= \sum_k \int \exp[ip(x_1 - y_1)] \frac{\exp\left[i \dfrac{p + 2\pi k}{L} \zeta\right]}{1 + 2\sum \left(1 - \cos \dfrac{p_i + 2\pi k_i}{L}\right)}$$

$$\times h\left(\frac{p + 2\pi k}{L}\right) \prod \frac{dp_i}{2\pi}$$

$$h\left(\frac{p + 2\pi k}{L}\right) = \prod_{i=1}^{2} \frac{(-1)^{k_i} \sin p_i/2}{L \sin(p_i + 2\pi k_i)/2L} \sim (-1)^{\sum k_i} \prod_{i=1}^{2} \frac{2 \sin p_i/2}{p_i + 2\pi k_i} \tag{B14}$$

where $x = Lx_1 + \zeta$ and the variables p_i in (B13) are replaced by $(p_i + 2\pi k_i)/L$ with $|p_i| \leqslant \pi$ and $k_i = 0, \pm 1, \ldots, \pm (L-1)/2$. It is enough to bound (B13) for $x_1 = y_1$. Then we can replace (B14) by $\prod_{i=1}^{2} (4\pi^2 k_i^2 + 1)^{-1}$ by the symmetry $p_i \to \pm p_i$. The denominator of (B13) is bounded from below by 1 and the derivative by x_μ (i.e., by ζ_μ) yields $\exp[i(p_\mu + 2\pi k_\mu)/L] - 1$. Then we use $k_\mu \to \pm k_\mu$ to see that the derivative of (B13) is of order $O(1/L)$. (The sum over k_i converges and then the contributions from small $|k|$ dominate.)

APPENDIX C. EXPLICIT FORM OF $\tilde{\mathscr{A}}_N$

Here we construct $\tilde{A}_1$ explicitly. The construction of $\tilde{A}_n$ is similar. For $M_0 \equiv \mathscr{T}_0^{\circ 2} + 2\mathscr{T}_0 \circ \mathscr{G}_1$, we put $M_{ij} \equiv P_i M_0 P_j$ so that

$$M_0 \equiv \begin{pmatrix} M_{00} M_{01} \\ M_{10} M_{11} \end{pmatrix}, \qquad \tilde{H}_0 \equiv M_0^{-1} \equiv \begin{pmatrix} \tilde{H}_{00} \tilde{H}_{01} \\ \tilde{H}_{10} \tilde{H}_{11} \end{pmatrix} \tag{C1}$$

namely

$$\tilde{H}_{00} = (M_{00} - M_{01} M_{11}^{-1} M_{10})^{-1}, \qquad \tilde{H}_{10} = -M_{11}^{-1} M_{10} \tilde{H}_{00}$$
$$\tilde{H}_{11} = (M_{11} - M_{10} M_{00}^{-1} M_{01})^{-1}, \qquad \tilde{H}_{01} = -M_{00}^{-1} M_{01} \tilde{H}_{11}$$

Since $P_i \mathscr{T}_0 P_0 = 0$ and $\mathscr{G}_1(x, y) - \beta_1 = O(1)$ for $|x - y| \ll m^{-1}$, we know that

(1) $P_0(\mathscr{T}_0 \circ \mathscr{G}_1) P_0$ is a positive operator bounded by a constant uniformly in β,

(2) $P_1(\mathscr{T}_0 \circ \mathscr{G}_1)\, P_1$ is a positive operator bounded from below by $O(\beta)$.

(3) $P_1(\mathscr{T}_0 \circ \mathscr{G}_1)\, P_0$ is an operator whose norm is bounded by $O(1)$.

Put $m = CM_0C^+$, namely put

$$m_{00}(x_1, y_1) = \frac{1}{L^4} \sum_{\zeta, \xi} M_0(Lx_1 + \zeta, Ly_1 + \xi)$$

and let $e = (1,..., 1)/\sqrt{L^2}$. Then $|e| = 1$, $(e, Qy) = 0$ for any $y \in R^{\Lambda\backslash\Lambda_1}$ and

$$M_{00}(x, y) = L^2 m_{00}([x/L], [y/L]) \circ (|e\rangle\langle e|),$$
$$M_{00}^{-1}(x, y) = L^{-2} m_{00}^{-1}([x/L], [y/L]) \circ (|e\rangle\langle e|)$$

Similar expressions also hold for $\tilde{H}_{00}$ and we have

$$M_{00}^{-1} C'^+ (C' M_{00}^{-1} C'^+)^{-1} = \tilde{H}_{00} C'^+ (C' \tilde{H}_{00} C'^+)^{-1} = \frac{1}{L^2} \delta_{[x/L], y}$$

Therefore

$$H_1 = C' \tilde{H}_0 C'^+ = \begin{pmatrix} C' \tilde{H}_{00} C'^+ & 0 \\ 0 & 0 \end{pmatrix} \tag{C2}$$

and we see that $(\tilde{A}_1)(x, y) = \tilde{H}_0 C'^+ H_1^{-1}(x, y)$ with $x \in \Lambda$ and $y \in \Lambda_1$ is given by

$$\begin{aligned}(\tilde{A}_1)(x, y) &= \frac{1}{L^2} \delta_{[x/L], y} - (M_{11}^{-1} M_{10})(x, Ly) \\ &= \frac{1}{L^2} \delta_{[x/L], y} - (M_{11}^{-1} M_{10} C^+)(x, y)\end{aligned} \tag{C3}$$

Let us consider the second factor of $\tilde{A}_1$. Since $M_{11}(x, y) = (1 - P_0)\, M_0 (1 - P_0) = Q^+ M_0 Q$ on $QR^{\Lambda\backslash L\Lambda_1}$, we have

$$M_{11}^{-1}(x, y) = (Q^+ M_0 Q)^{-1} \sim (2\beta Q^+ Q \Gamma_0 Q^+ Q)^{-1} \tag{C4}$$

on $QR^{\Lambda\backslash L\Lambda_1}$. Therefore

$$M_{11}^{-1} \sim (2\beta Q^+ Q \Gamma_0 Q^+ Q)^{-1} = \frac{1}{2\beta} (Q^+ Q)^{-1} [Q^+(-\Delta + m^2)\, Q] (Q^+ Q)^{-1} \tag{C5}$$

where $(Q^+Q)^{-1}$: $R^{\Lambda\backslash L\Lambda_1} \to R^{\Lambda\backslash L\Lambda_1}$ is given by $(Q^+Q)^{-1} = 1 - \mathscr{P} + \frac{1}{L^2} \mathscr{P}$ with $\mathscr{P}$ being the projection onto the set of the block-wise $(\square_{Lx} \backslash \{Lx\})$ constant functions. Thus we obtain

$$(Q^+Q\Gamma_0Q^+Q)^{-1}_{xy} = (-\Delta+m^2)_{x,y} + \frac{1}{L^2}(\delta_{|Lx_1-y|,1} + \delta_{|Ly_1-x|,1}) - \left(1-\frac{1}{L^2}\right)$$
$$\times [\mathscr{P}(-\Delta+m^2) + (-\Delta+m^2)\,\mathscr{P}]_{xy} + \delta\mathscr{L}_{xy} \tag{C6a}$$

$$|\delta\mathscr{L}(x, y)| = \left(1-\frac{1}{L^2}\right)^2 [\mathscr{P}(-\Delta+m^2)\,\mathscr{P}]_{xy}$$
$$+\left(\frac{4+m^2}{L^4} - \frac{8}{L^2(L^2-1)} + \frac{2}{L^6}\right)\delta_{x_1, y_1} \tag{C6b}$$

where $x, y \in \Lambda \backslash L\Lambda_1$ and $x_1 \equiv [x/L] \in \Lambda_1$. To rewrite this as the map from $QR^{\Lambda\backslash L\Lambda_1}$ into $QR^{\Lambda\backslash L\Lambda_1}$, we multiply Q to the left and Q^+ to the right.

APPENDIX D. SOME INTEGRALS

We first consider the one dimensional case. Using

$$\exp[-\alpha\,|x|] = \int \frac{2\alpha e^{ipx}}{p^2+\alpha^2}\,\frac{dp}{2\pi}$$

we have

$$\int \exp\left(-\alpha \sum_{i=0}^{n} \left|\frac{\xi_{i+1}}{L} - \xi_i\right|\right) \prod_{1}^{n} d\xi_i$$
$$= (2\alpha)^{n+1} \int \exp\left[ip\left(\frac{x}{L^{n+1}} - y\right)\right]\left(\prod_{k=0}^{n} \frac{L^{2k}}{p^2+\alpha^2 L^{2k}}\right)\frac{dp}{2\pi} \tag{D1}$$

Then if $x/L^{n+1} - y > 0$, the contributions come from the $n+1$ poles $p = iL^k\alpha$, $k = 0, 1, \ldots, n$. The contribution from $p = i\alpha$ gives the dominant part and the sum of other contributions converges (even if $y = x/L^{n+1}$) and is less than that. Then we have the formula

$$\text{RHS of Eq. (D1)} = \left(1+O\left(\frac{1}{L}\right)\right)(2\alpha)^{n+1} \exp\left[-\alpha\left|\frac{x}{L^{n+1}} - y\right|\right]$$
$$\times \left(\frac{1}{2\alpha\cdot\alpha^{2n}} \prod_{k=1}^{n} \frac{L^{2k}}{L^{2k}-1}\right)$$
$$= \left(1+O\left(\frac{1}{L}\right)\right)\left(\frac{2}{\alpha}\right)^n \exp\left[-\alpha\left|\frac{x}{L^{n+1}} - y\right|\right]$$

We note that contribution from the residue $p = i\alpha L^k$ get the additional multiplicative factor of order $O(L^{-k})$. In the two dimensional case, we use the identity

$$\exp[-\alpha\,|x|] = 2\pi\alpha \int \frac{e^{ipx}}{(p^2+\alpha^2)^{3/2}} \frac{d^2p}{(2\pi)^2}$$

Then (3.11) follows from this in the same way.

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